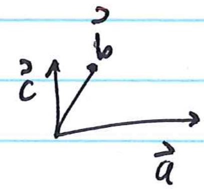
152 Lecture 4

Last day: determinant, cross prod etc.

$\left|\det \begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix}\right|$ is the volume of parallelopiped



Ex Find the volume of the parallelopided with sides $\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = 0 \cdot \det(\quad) - 0 \det(\quad) + 1 \det \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= -2$$

$\det \begin{bmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{bmatrix} = 0 \iff$ $\vec{a}, \vec{b}, \vec{c}$ lie in common plane (~~not~~ don't have 3 "different directions")

Ex. Recall: $\vec{a} \times \vec{b}$ orthog to $\vec{a}$ (and $\vec{b}$)

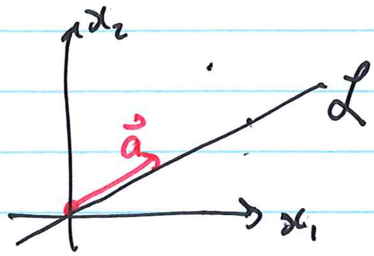
Prove: triple product $\vec{a} \cdot (\vec{a} \times \vec{b}) = \det \begin{bmatrix} - & \vec{a} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{bmatrix}$

$= \text{volume} = 0$

Points, Lines and Planes §2.5

The **point** $\vec{x} = (x_1, x_2)$ (or (x, y)) means the point at the head of the vector $\vec{x}$ with tail at origin.

Lines through the origin (in 2D)



Every point $\vec{x}$ on the line $\mathcal{L}$ is a scalar mult. of direction $\vec{a} \neq \vec{0}$

$$\boxed{\vec{x} = s\vec{a}}$$

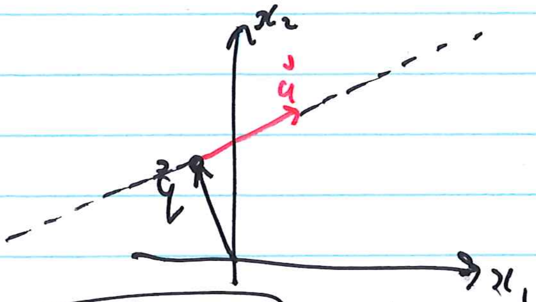
set of points
$\hookrightarrow \mathcal{L} = \{\vec{x} \mid \vec{x} = s\vec{a} \text{ for some } s \in \mathbb{R}\}$

2D: line through point $\vec{q}$

arbitrary pt on line.

$$\boxed{\vec{x} = \vec{q} + s\vec{a}}$$

parameter.



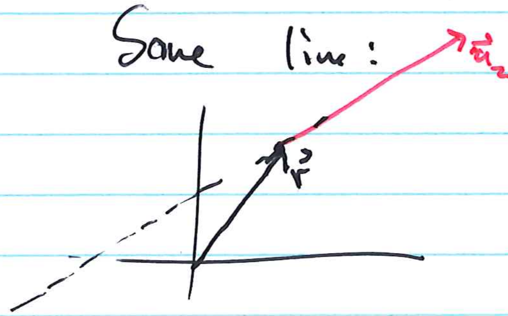
parametric eqn for a line

not unique

what about "$y = mx + b$"?

later

Same line:



$\vec{x} = \vec{r} + t\vec{a}_2$

Lines in 3D : parametric form:

$$\vec{x} = \vec{q} + s\vec{a}$$

arbitrary pt in $\mathbb{R}^3$ dir in $\mathbb{R}^3$

Ex $\mathcal{L}$ passes through $(1,1,1)$ and $(2,3,7)$, find the parametric eqn.

$$\vec{a} = (2,3,7) - (1,1,1) = (1,2,6)$$

$$\vec{q} = (1,1,1) \quad \text{or} \quad (2,3,7)$$

$$\vec{x} = (1,1,1) + s(1,2,6), \quad s \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+s \\ 1+2s \\ 1+6s \end{bmatrix}$$

Plane in 3D

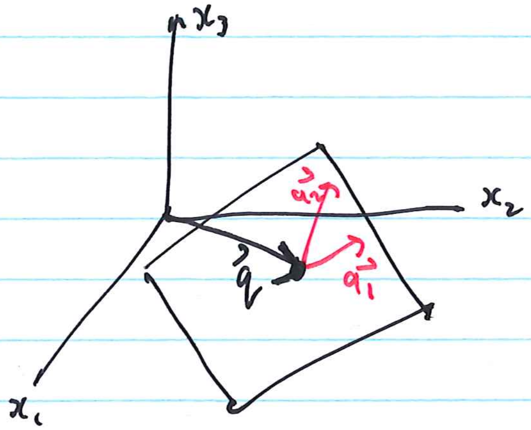
← 2 dimensional, two parameters (?)

$$\boxed{\vec{x} = \vec{q} + \underbrace{s\vec{a}_1 + t\vec{a}_2}}$$

~~every~~ any point on a plane can be found by starting at...

an arbitrary pt on the plane plus...

linear combination of two different directions $\vec{a}_1 \neq 0$ and $\vec{a}_2 \neq 0$



Ex pts $(1,1,1)$, $(2,3,7)$, $(0,2,0)$ on ~~plane~~ plane $\mathbb{P}$, write in parametric form.

$\vec{a}_1 = (2,3,7) - (1,1,1) = (1,2,6)$

$\vec{a}_2 = (0,2,0) - (1,1,1) = (-1,1,-1)$

not multiples

$$\mathbb{P} \text{ is } \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Aside "different directions"?

$$\begin{cases} \vec{a}_1 \neq m\vec{a}_2 & \text{for any } m \\ \vec{a}_1 \neq \vec{0}, \quad \vec{a}_2 \neq \vec{0} \end{cases}$$

if $\vec{a}_1 = m\vec{a}_2$ for some $m \in \mathbb{R}$, we say $\vec{a}_1$ and $\vec{a}_2$ are collinear or linearly dependent

(includes $\vec{a}_1 = 0$ special case)

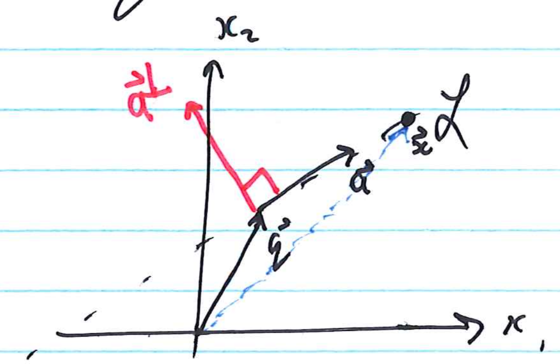
if $\vec{a}_1$ and $\vec{a}_2$ are linearly independent ~~means~~ (not collinear)

is the formal way to say "different directions"

more on this later.

aka "Implicit"

"Equation form" of a line in 2D



Recall $\vec{a}^{\perp} := (-a_2, a_1)$

$$\boxed{(\vec{x} - \vec{q}) \cdot \vec{a}^{\perp} = 0} \iff \text{point } \vec{x} \text{ is on } \mathcal{L}$$

~~plane~~

or

$$\vec{x} \cdot \vec{a}^{\perp} = \vec{q} \cdot \vec{a}^{\perp} = d, \text{ scalar}$$

$$\Rightarrow \vec{a}_1^{\perp} x_1 + \vec{a}_2^{\perp} x_2 = d \qquad (*)$$

$$\Rightarrow \boxed{a_1 x_2 - a_2 x_1 = d} \qquad (**)$$

→ like $y = mx + b$

equation form of the line (in 2D)

~~Ex is (5,5) on the line~~

Ex line through $(1,2)$ and $(3,3)$, find equation form.

$$\vec{a} = (2, 1)$$
$$\vec{a}^{\perp} = (-1, 2) \qquad \text{take } \vec{q} = (1, 2)$$

$$(*) \Rightarrow -x_1 + 2x_2 = d = \vec{a}^{\perp} \cdot \vec{q} = -1 + 4 = 3$$

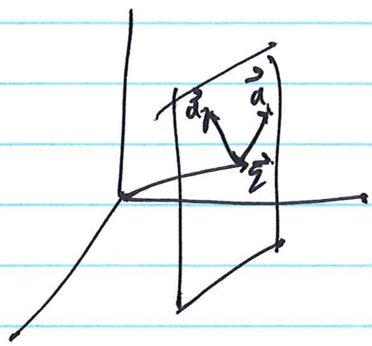
$$\boxed{-x_1 + 2x_2 = 3}$$

Ex is $(5,5)$ on the previous line?

plug into LHS of eqn form:

$$-5+2(5) = 5 \neq 3$$

Equation form of a plane (in 3D)



Plane $\mathbb{P}$ has normal vector $\vec{n}$ orthogonal to the plane

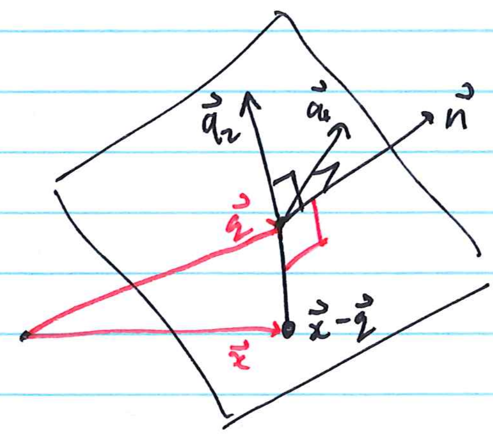
↳ $\vec{n} = \vec{a}_1 \times \vec{a}_2$ (or multiple)

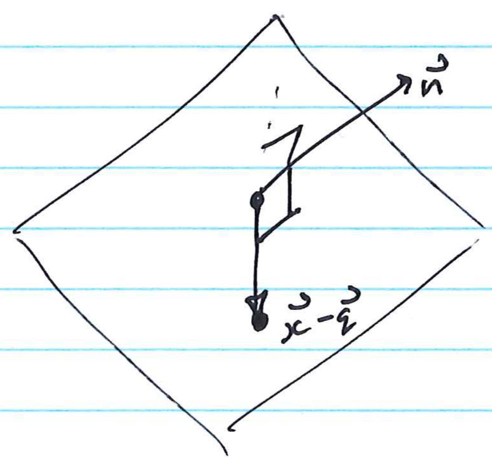
need $\vec{a}_1, \vec{a}_2$ linearly independent!

Any point $\vec{x}$ in plane satisfies

$$\boxed{(\vec{x} - \vec{q}) \cdot \vec{n} = 0}$$

eqn form of a plane.





or

$$\vec{x} \cdot \vec{n} = \vec{q} \cdot \vec{n} = d, \text{ scalar}$$

linear eqn in 3 variables

$$\boxed{n_1 x_1 + n_2 x_2 + n_3 x_3 = d}$$

Ex $\mathbb{P}$ containing 3 pts (as before)
$(1,1,1)$, $(2,3,7)$ and $(0,2,0)$

from before $\vec{a}_1 = (1,2,6)$, $\vec{a}_2 = (-1,1,-1)$

$$\vec{n} := \vec{a}_1 \times \vec{a}_2 = \dots = (-8,-5,3)$$

$$d = \vec{n} \cdot (1,1,1) = -8-5+3 = -10$$

so $\boxed{-8x_1 - 5x_2 + 3x_3 = -10}$

Ex find the normal vector of the plane
$3x + 4y - 2z = 1$

$\longrightarrow$ $\vec{n} = (3,4,-2)$ (not unique)

Ex are these planes parallel?

$$3x + 4y - 2z = 2$$
$$6x + 8y - 4z = 0$$
?

yes b/c $(6,8,-4) = 2(3,4,-2)$

Equation form of line in 3D

not ~~an~~ one linear in (x, y, z) . . .

Instead we intersect two planes.

Ex $\mathcal{L}: \vec{x} = \underset{\vec{q}}{(1, 1, 1)} + s\underset{\uparrow \vec{a}}{(1, 2, 6)}$

Need two linearly independent vectors $\vec{b}_1$ and $\vec{b}_2$ both $\perp$ to $\vec{a}$

trick: $\vec{b}_1 = \left[[1, 2]^\perp \; 0 \right] = \left[-2 \; 1 \; 0 \right]$

then $\vec{b}_1 \cdot \vec{a} = -2 + 2 + 0 = 0$

$\vec{b}_2 = \cancel{\vec{a} \times \vec{b}} = \vec{a} \times \vec{b}_1 = [-6, -12, 5]$

$\hookrightarrow$ two planes: $d_1 = \vec{q} \cdot \vec{b}_1 = -1$
$d_2 = \vec{q} \cdot \vec{b}_2 = -13$

$$\boxed{\begin{aligned} \vec{b}_1 \cdot \vec{x} &= -1 \\ \vec{b}_2 \cdot \vec{x} &= -13 \end{aligned}} \Rightarrow \boxed{\begin{aligned} -2x_1 + x_2 &= -1 \\ -6x_1 - 12x_2 + 5x_3 &= -13 \end{aligned}}$$

- simultaneous system of 2 equations
- line is the sol'n of both eqns simultaneously.