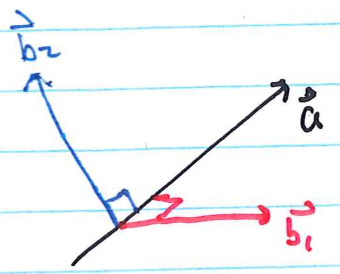
152 Lecture 5

Last day: eqn form of plane (in 3D):

$$\left\{\begin{array}{c} \vec{n}\cdot\vec{x} = d \\ \text{or} \\ \vec{n}\cdot(\vec{x}-\vec{q}) = 0 \\ \text{or} \\ \boxed{n_1x_1 + n_2x_2 + n_3x_3 = d} \end{array}\right.$$

Eqn form of line in 3D → intersect two planes

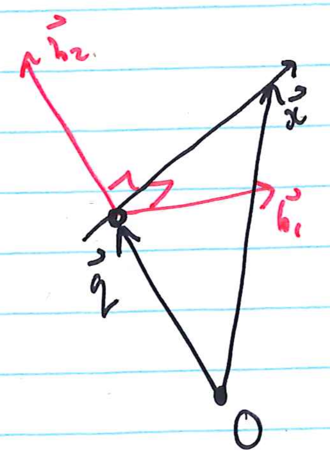
> line's direction vector $\vec{a}$ must be orthog to the normals $\vec{b}_1$ and $\vec{b}_2$ of two planes



Eg. given parametric $\vec{x} = \vec{q} + s\vec{a}$

↓

find $\vec{b}_1$ and $\vec{b}_2$ (somehow)



need: $$\left.\begin{array}{l} \vec{b}_1\cdot(\vec{x}-\vec{q}) = 0 \\ \vec{b}_2\cdot(\vec{x}-\vec{q}) = 0 \end{array}\right\}$$ 2 planes ⇓ $\vec{x}$ on both

equiv:

$$\boxed{\begin{array}{l} b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 = d_1 \\ b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 = d_2 \end{array}}$$

2 eqns in 3 unknowns → cannot solve for just one answer

of course! b/c solution is the line

Note: converting from eqn form to parametric form. $\Rightarrow$ defer to Chapter 3.
↳ how to solve systems in general

Intro to linear systems (in 2D/3D)

1 linear eqn in two unknowns: $b_{1,1}x_1 + b_{1,2}x_2 = c_1$

same

$$\boxed{\vec{b}_1 \cdot \vec{x} = c_1}$$

coefficients unknowns data

This ~~was~~ is the eqn form of a line in 2D. So sol'n is a line.

Ex $2x + 4y = 10$

Choose $x = s$
then $4y = -2s + 10$
$y = -\frac{1}{2}s + \cancel{\frac{5}{2}}\, 10/4$

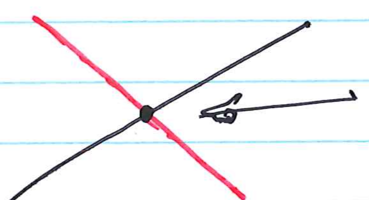
parametric form $\longrightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ -\frac{1}{2}s + \frac{10}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{10}{4} \end{bmatrix} + s\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

- we found infinitely many sol'ns
- $\vec{b}_1 = \vec{0}$? then $\begin{cases} \text{if } c_1 = 0 & \text{then } \underline{\underline{\text{any}}}\ \vec{x} \text{ is sol'n.} \\ \text{if } c_1 \neq 0 & \text{no sol'n.} \end{cases}$
- We found: infinitely many sol'ns / no sol'ns
  <u>later</u>: <u>or</u> exactly one sol'n.

<u>2 eqns in 2 unknowns</u>

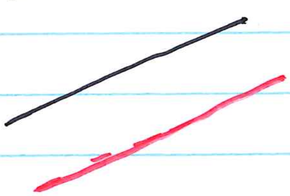
$$\left.\begin{aligned} \vec{b}_1 \cdot \vec{x} &= c_1 \\ \vec{b}_2 \cdot \vec{x} &= c_2 \end{aligned}\right\} \quad \begin{aligned} b_{1,1}x_1 + b_{1,2}x_2 &= c_1 \\ b_{2,1}x_1 + b_{2,2}x_2 &= c_2 \end{aligned}$$

~~Generic~~ intersection of two lines



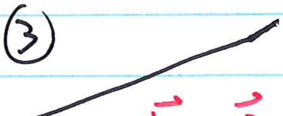
<u>unique</u> sol'n at pt of intersection

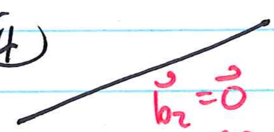
<u>Unless</u>

(1)

no sol'n

(2)

infinitely many sol'ns
every x on the line

(3)

$\vec{b}_2 = \vec{0}$
$c_2 = 0$
inf many sol'ns

(4)

$\vec{b}_2 = \vec{0}$
$c_2 \neq 0$
no sol'n

(5) $\vec{b}_1 = \vec{0}$, $\vec{b}_2 = \vec{0}$
$c_1 = c_2 = 0$ — inf many

(6) ″, $c_1 \neq 0$ — no sol'n.

all these "special cases" have $\det \begin{bmatrix} - \vec{b}_1 - \\ - \vec{b}_2 - \end{bmatrix} = 0$

General — For any linear system

Major goal of Math 152

(any number m of eqns, in any number n unknowns),

$$b_{1,1}x_1 + b_{1,2}x_2 + \dots + b_{1,n}x_n = c_1$$
$$\vdots$$
$$b_{m,1}x_1 + b_{m,2}x_2 + \dots + b_{m,n}x_n = c_m$$

we want:

- does it have 0, 1 or ∞ sol'ns?
- an algorithm to find those sol'ns.

3 eqns in 3 unknowns $\quad \vec{x} = (x_1, x_2, x_3)$

$$\vec{b}_1 \cdot \vec{x} = c_1$$
$$\vec{b}_2 \cdot \vec{x} = c_2$$
$$\vec{b}_3 \cdot \vec{x} = c_3$$

- Unique sol'n if and only if $\det \begin{bmatrix} - \vec{b}_1 - \\ - \vec{b}_2 - \\ - \vec{b}_3 - \end{bmatrix} \neq 0$

  1 pt in $\mathbb{R}^3$

- Intersection of 3 planes w/ normals $\vec{b}_1, \vec{b}_2, \vec{b}_3$

$\boxed{\det = 0?}$ $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ all live in one plane.
(recall parallelopiped)

e.g., one is a linear combination of others.

Ex planes could intersect in a line.

sol'ns $\vec{x} = \vec{q} + s\vec{a}$

1-dimensional set of sol'ns

Ex all planes same

sol'ns are the plane:

$\vec{x} = \vec{q} + s\vec{a}_1 + t\vec{a}_2$

2-dimensional set of sol'ns.

Ex two planes (or more) parallel but non-intersecting

no sol'ns

(empty set of sol'ns.

Ex etc.

Ex does this system have a single unique sol'n?

$$\boxed{\begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_3 &= 0 \\ x_1 + x_3 &= 2 \end{aligned}}$$

$$\det\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \ldots = 2 \neq 0$$

yes

Ex (same)

$$\boxed{\begin{aligned} x + y &= 1 \\ y + z &= 0 \end{aligned}}$$

no, b/c two non parallel planes so intersection is a line

Ex (same)

$$\boxed{\begin{aligned} x + y &= 1 && (i) \\ y + z &= 0 && (ii) \\ x + 2y + z &= 2 && (iii) \end{aligned}}$$

$\det\left(\quad \right) = 0$ no

Note $(i) + (ii)$

$\Rightarrow x + 2y + z = 1 \neq 2$

no sol'ns $\leftarrow$ inconsistent data.

(§2.6) Spans, etc. (bit more abstract)

defn: $s_1 \vec{a}_1 + s_2 \vec{a}_2$ $(+ s_3 \vec{a}_3 + \ldots)$ is called a linear combination of the vectors $\vec{a}_1, \vec{a}_2, \ldots$

defn span$\{\vec{a}, \vec{b}\}$ is all possible linear combinations (every $s_1 \in \mathbb{R}$, every $s_2 \in \mathbb{R}$ etc.

Ex plane in parametric form:

$$\vec{x} = \vec{q} + \text{span}\{\vec{a}_1, \vec{a}_2\}$$

line in parametric form:

$$\vec{x} = \vec{q} + \text{span}\{\vec{a}\}$$

defn a set of vectors is linearly dependant if some linear combination is 0:

$$s_1 \vec{a}_1 + s_2 \vec{a}_2 + \ldots + s_n \vec{a}_n = 0$$

for $s_1, s_2, \ldots, s_n$ not all zero.

defn linearly independent set of vectors if that linear combination is zero only when all $s_i = 0$.

Ex $s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \vec{0}$?

Ex if $\vec{a}_1, \vec{a}_2, \vec{a}_3$ linearly dep. then they lie on a plane.

Consider a collection of vectors

$$\mathcal{C} = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$$

$\in \mathbb{R}^n$

and think about span $\mathcal{C}$

- if $m \le n$ and $\mathcal{C}$ is l.i. (linearly indep.) then span $\mathcal{C}$ is "m-dimensional subspace of $\mathbb{R}^n$"

  e.g. 2-dimensional plane in 3D

- if $m > n$ cannot be l.i.

- how do we test l.i.?

  $n = m = 3 \rightarrow$ use det zero or nonzero

  $m = 2$, any $n \rightarrow$ check collinear (with componentwise division)

Ex span $\{(2,3,4,5), (2,4,5,7)\}$

l.i.? $\frac{2}{2} = 1$ but $\frac{4}{3} \neq 1$

not collinear so linearly dependent.

Thus span is two-dimensional subspace of $\mathbb{R}^4$

In general n, m : ? — <u>later.</u>