152 Lecture 6

Last day: span$\{\vec{a}, \vec{b}\}$ = all possible linear combinations of $\vec{a}$ and $\vec{b}$

Ex write $[1,1]$ as linear combination of $\vec{a} = [2,3]$ and $\vec{b} = [1,2]$

$$[1,1] = s[2,3] + t[1,2]$$

[back up: is it even possible? note $\vec{a}, \vec{b}$ not collinear $\Rightarrow$ span$\{\vec{a}, \vec{b}\} = \mathbb{R}^2$

if $\vec{a}, \vec{b}$ collinear span$\{\vec{a}, \vec{b}\}$ = span$\{\vec{a}\}$ $= s\vec{a}$]

today

A:
$$\begin{aligned} 2s + t &= 1 \\ 3s + 2t &= 1 \end{aligned}$$
a linear system

solve (somehow) ... $\boxed{s=1}$ and $\boxed{t=-1}$

So $[1,1] = [2,3] - [1,2]$.

Note: "1" and "−1" are coordinates in $\vec{a}$, $\vec{b}$ directions.

Ex Can $[1,1,1]$ be written as a linear comb. of $[1,-2,1]$ and $[1,0,1]$? 3

↳ and what is it?

Algebra: $[1,1,1] = s[1,-2,1] + t[1,0,1]$ 3

[Geom: is $[1,1,1]$ on the plane given by (through origin)]

$$\begin{aligned} s+t &= 1 \\ -2s+0 &= 1 \\ s+t &= 1 \end{aligned} \qquad t = 3/2 \Rightarrow s = -1/2 \checkmark$$

$$\left.\begin{aligned} s+t &= 1 \\ -2s &= 1 \end{aligned}\right\} \begin{aligned} t &= 3/2 \\ s &= -1/2 \end{aligned}$$

$$s + 3t = 1 \quad \times$$

b/c $-1/2 + 9/2 = 4 \neq 1$

3 eqns, 2 unknowns

Ex Fred and Louise together have 8 bitcoin. Louise has 2 more bitcoin than Fred. How many does each have?

Let x_1 be # of Fred's bitcoin

x_2 " " " Louise's bitcoin.

Model: $$\begin{aligned} x_1 + x_2 &= 8 \\ -x_1 + x_2 &= 2 \end{aligned}$$ System of eqns

Solve: add, $2x_2 = 10$, $x_2 = 5$, $x_1 = 3$

Fred has 3 bitcoin, Louise has 5 bitcoin

Solving Linear Systems

m eqns
n unknowns

specific example:

$$(*)\quad \begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 1 \quad (1)\\ x_2 + x_3 &= 0 \quad (2)\\ x_1 \phantom{{}+x_2} + x_3 &= 2 \quad (3) \end{aligned}$$

the Augmented Matrix

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\text{coeff}} \left|\, \underbrace{\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}}_{\text{data}} \right.$$

Three operations that leave sol'n unchanged.

① interchange two eqns (or swap)

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 1 \\ x_1 \phantom{{}+x_2} + x_3 &= 2 \qquad (2)\leftrightarrow(3) \\ x_2 + x_3 &= 0 \end{aligned}$$

① interchange two rows

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

equivalent: lin sys has same soln.

② mult. eqn by non-zero number

$$\begin{aligned} 3x_1 + 3x_2 \phantom{{}+x_3} &= 3 \qquad (1)\leftarrow 3(1) \\ x_1 \phantom{{}+x_2} + x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

②

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

③ add a multiple of one equation to another.

$$\begin{aligned} 3x_1 + 3x_2 \phantom{{}+x_3} &= 3 \\ 4x_1 + 3x_2 + x_3 &= 5 \qquad (2)\leftarrow(2)+1(1) \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 0 & 3 \\ 4 & 3 & 1 & 5 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

This new system has same sol'ns as (*) b/c each operation is reversible

Systems that are *easiest* to solve

reduced row echelon form →

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

so $x_3 = 5$, $x_2 = 2$, $x_1 = 9$

Easier

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

← echelon form

Back substitution start with last eqn, work upwards

$3x_3 = 3 \Rightarrow x_3 = 1$

$2x_2 = 1 - 5x_1 = 1 - 5(1) = 1 - 5 = -4$

$\Rightarrow x_2 = -2$

$\cancel{x_1 = 6 - x_2 - x_3 = 6-(4)-(1) = 9}$

$x_1 = 6 - x_2 - x_3 = 6 - (-2) - 1 = 7$

Ex on (*)

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \underset{③ \leftarrow ③ - ①}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \underset{③ \leftarrow ③ + ②}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

nice!

eliminate.

echelon form

Back sub: $2x_3 = 1$, $x_3 = 1/2$

$x_2 = 0 - x_3 = -1/2$

$x_1 = 1 - x_2 = 1 - -1/2 = 3/2.$

This is called Gaussian Elimination and computers spend a lot of their time doing it!

Algorithm: sequence of steps to solve a problem.

idea behind computer code.

GE Algorithm : input augmented matrix
$n+1$ ~~n+1~~ columns
m rows

a_{ij} is entry in the ith row jth col.

$\boxed{0}$ start with $i=1$ (row index) and $j=1$ (column index)

repeat $\boxed{1}$ if $a_{ij} \neq 0$ go to $\boxed{2}$
else find row k where $a_{kj} \neq 0$
swap $\text{row}_i \leftrightarrow \text{row}_k$ go to $\boxed{1}$
else can't find (i.e., column is all zero)
$j \leftarrow j+1$ go to $\boxed{1}$

$\boxed{2}$ for rows $k = i+1, \ldots, m$
$$\text{row}_k \leftarrow \text{row}_k - \left(\frac{a_{kj}}{a_{ij}}\right) \text{row}_i$$
$\neq 0$ (eliminate x_j from eqn k)

$\boxed{3}$ $i \leftarrow i+1$, $j \leftarrow j+1$

~~unit~~
until $i = n$ or $j = m$

Ex:
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

<u>def'n</u>: <u>leading entry</u> of a row is the first nonzero entry from left.

<u>GE</u> gives us <u>echelon form</u>!

1. all not-all-zero rows above all-zero rows
2. each leading entry has zeros in the column below.

columns already zero in step $\boxed{1}$

$$\left[\begin{array}{ccccccc|c} \boxed{x} & x & x & x & x & x & x & x \\ 0 & \boxed{x} & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & \boxed{x} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \boxed{x} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Big\} \text{ all-zero rows}$$

reduced row echelon form (<u>rref</u>)

① and ② and

3. all leading entries one!
4. all entries above leading entry are 0

$$\sim \begin{bmatrix} \boxed{1} & 0 & x & x & 0 & 0 & x & x \\ 0 & \boxed{1} & x & x & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Note</u>: find rref from echelon form is done with more row operations

in practice, almost same as <u>back sub</u>

Questions – how to solve the system?

Ex
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right] \underset{(3)-(1)}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \begin{array}{r} \\ \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

back sub: $0x_1 + 0x_2 + 0x_3 = 1$

OH SNAP!

no sol'n

No sol'ns after GE: signature

$$\left[\begin{array}{ccc|c} & \sim & & \\ 0 & \cdots & 0 & x \end{array}\right] \longrightarrow \text{non zero}$$

zero row in coeff

} no sol'ns

Next day: all zero row (w/ 0 data).

$\Downarrow$ infinitely many sol'n's