152 Lecture 6

Last day: $\text{span}\{\vec{a}, \vec{b}\}$ = all possible linear combinations of $\vec{a}$ and $\vec{b}$

Ex write $[1,1]$ as linear combination of $\vec{a} = [2,3]$ and $\vec{b} = [1,2]$

$$[1,1] = s[2,3] + t[1,2]$$

[back up: is it even possible? note $\vec{a}, \vec{b}$ not collinear $\Rightarrow \text{span}\{\vec{a}, \vec{b}\} = \mathbb{R}^2$

if $\vec{a}, \vec{b}$ collinear $\text{span}\{\vec{a}, \vec{b}\} = \text{span}\{\vec{a}\} = s\vec{a}$]

A:
$$2s + t = 1$$
$$3s + 2t = 1$$

a linear system

today

solve somehow ... $\boxed{s = 1}$ and $\boxed{t = -1}$

So $[1,1] = [2,3] - [1,2]$.

Note: "1" and "-1" are coordinates in $\vec{a}, \vec{b}$ directions

Ex Can [1, 1, 1] be written as a <u>linear comb.</u> of [1, -2, 1] and [1, 0, 1 3]?
↳ and what is it?

Algebra: $[1,1,1] = s[1,-2,1] + t[1,0,1\ 3]$

[Geom: is [1, 1, 1] on the plane given by (through origin)]

$$\begin{aligned} s + t &= 1 \\ -2s + 0 &= 1 \\ s + t &= 1 \end{aligned} \qquad t = 3/2 \quad \Rightarrow s = -1/2 \quad ✓$$

$$\left.\begin{aligned} s + t &= 1 \\ -2s &= 1 \end{aligned}\right\} \begin{aligned} t &= 3/2 \\ s &= -1/2 \end{aligned}$$
$$s + 3t = 1 \quad \times$$

b/c $-\frac{1}{2} + \frac{9}{2} = 4 \neq 1$

3 equns, 2 unknowns

Ex Fred and Louise together have 8 bitcoin. Louise has 2 more bitcoin than Fred. How many does each have?

let x_1 be # of Fred's bitcoin
x_2 " " " Louise's bitcoin.

<u>Model</u>:
$$\begin{aligned} x_1 + x_2 &= 8 \\ -x_1 + x_2 &= 2 \end{aligned}$$

System of equns

Solve: add, $2x_2 = 10$, $x_2 = 5$, $x_1 = 3$

Fred has 3 bitcoin, Louise has 5 bitcoin

Solving Linear Systems

m eqns
n unknowns

specific example:

$$(*)\quad \begin{aligned} x_1 + x_2 \quad &= 1 \qquad (1)\\ x_2 + x_3 &= 0 \qquad (2)\\ x_1 \quad + x_3 &= 2 \qquad (3)\end{aligned}$$

the Augmented Matrix

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 1\\ 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 2 \end{array}\right]$$

$\underbrace{}_{\text{coeff}}$ ↑ data

Three operations that leave sol'n unchanged.

① interchange or swap two eqns

$$\begin{aligned} x_1 + x_2 \quad &= 1\\ x_1 \quad + x_3 &= 2 \qquad (2)\leftrightarrow(3)\\ x_2 + x_3 &= 0 \end{aligned}$$

① interchange two rows

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 2\\ 0 & 1 & 1 & 0 \end{array}\right]$$

equivalent: lin sys has same sol'n.

② mult. eqn by non-zero number

$$\begin{aligned} 3x_1 + 3x_2 \quad &= 3 \qquad (1)\leftarrow 3(1)\\ x_1 \quad + x_3 &= 2\\ x_2 + x_3 &= 0 \end{aligned}$$

②

$$\sim \left[\begin{array}{ccc:c} 3 & 3 & 0 & 3\\ 1 & 0 & 1 & 2\\ 0 & 1 & 1 & 0 \end{array}\right]$$

③ add a multiple of one equation to another.

$$\begin{aligned} 3x_1 + 3x_2 \quad &= 3\\ 4x_1 + 3x_2 + x_3 &= 5 \qquad (2)\leftarrow(2)+1(1)\\ x_2 + x_3 &= 0 \end{aligned}$$

$$\sim \left[\begin{array}{ccc:c} 3 & 3 & 0 & 3\\ 4 & 3 & 1 & 5\\ 0 & 1 & 1 & 0 \end{array}\right]$$

This new system has same sol'ns as (*) b/c each operation is reversible

Systems that are easiest to solve

reduced row echelon form →

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

so $x_3 = 5, \quad x_2 = 2, \quad x_1 = 9$

Easier

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

← echelon form

Back substitution start with last eqn, work upwards

$$3x_3 = 3 \Rightarrow x_3 = 1$$

$$2x_2 = 1 - 5x_1 = 1 - 5(1) = 1 - 5 = -4$$

$$\Rightarrow x_2 = -2$$

$$\cancel{x_1 = 6 - x_2 - x_3 = 6 - (4) - (1) = 9}$$

$$x_1 = 6 - x_2 - x_3 = 6 - (-2) - 1 = 7$$

Ex on (*)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \underset{③ \leftarrow ③ - ①}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \; ③ \leftarrow ③ + ②$$

nice!

eliminate.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

echelon form

Back sub: $2x_3 = 1, \; x_3 = 1/2$

$x_2 = 0 - x_3 = -1/2$

$x_1 = 1 - x_2 = 1 - -1/2 = 3/2.$

This is called <u>Gaussian Elimination</u> and computers spend <u>a lot</u> of their time doing it!

<u>Algorithm:</u> sequence of steps to solve a problem.

idea behind computer code.

GE Algorithm : input augmented matrix
$n+1$ ~~n+1~~ columns
m rows

a_{ij} is entry in the i^{th} row j^{th} col.

row index, column index

$\boxed{0}$ start with $i=1$ and $j=1$

repeat $\boxed{1}$ if $a_{ij} \neq 0$ go to $\boxed{2}$
else find row k where $a_{kj} \neq 0$
swap $row_i \leftrightarrow row_k$ go to $\boxed{1}$
else can't find (i.e., column is all zero)
$j \leftarrow j+1$ go to $\boxed{1}$

$\boxed{2}$ for rows $k = i+1, \ldots, m$

$$row_k \leftarrow row_k - \left(\frac{a_{kj}}{a_{ij}}\right) row_i$$

$\neq 0$ (eliminate x_j from eqn k

$\boxed{3}$ $i \leftarrow i+1$, $j \leftarrow j+1$

~~unit~~
until $\boxed{i = n \text{ or } j = m}$ ← update: typo here, should be $i = m$ or $j = n$

Ex:
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

<u>def'n</u>: <u>leading entry</u> of a row is the first nonzero entry from left

<u>GE</u> gives us <u>echelon form</u>:

① all not-all-zero rows above all-zero rows

② each leading entry has zeros in the column below.

columns already zero in step $\boxed{1}$

$$\left[\begin{array}{ccccccc|c} \boxed{x} & x & x & x & x & x & x & x \\ 0 & \boxed{x} & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & \boxed{x} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \boxed{x} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Big\} \text{ all-zero rows}$$

reduced row echelon form (<u>rref</u>)

① and ② and

③ all leading entries one!

④ all entries above leading entry are 0

$$\sim \begin{bmatrix} \boxed{1} & 0 & x & x & 0 & 0 & x & x \\ 0 & \boxed{1} & x & x & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Note</u>: find rref from echelon form is done with more row operations

in practice, almost same as <u>back sub</u>

Questions – how to solve the system?

Ex

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right] \underset{(3)-(1)}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

back sub: $0x_1 + 0x_2 + 0x_3 = 1$

OH SNAP!

no sol'n

No sol'ns after GE: signature

$$\left[\begin{array}{ccc|c} & \sim & & \\ 0 & \cdots & 0 & x \end{array}\right] \quad x \longrightarrow \text{non zero}$$

zero row in coeff

} no sol'ns

Next day: all zero row (w/ 0 data).

$\Downarrow$ infinitely many sol'ns