Math 152 lecture 7

Recall we've solving m equns in n unknowns

$$\begin{aligned} b_{1,1}x_1 + b_{1,2}x_2 + \dots + b_{1,n}x_n &= c_1 \\ &\vdots \\ b_{m,1}x_1 + b_{m,2}x_2 + \dots + b_{m,n}x_n &= c_m \end{aligned}$$

Augmented matrix :

$$\underbrace{\left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{b_{1,1} \quad b_{1,2} \quad \dots \quad b_{1,n}}}^{\text{matrix of coeffs}}} & \text{"RHS"} \\ b_{1,1} & b_{1,2} & \dots & b_{1,n} & c_1 \\ \vdots & & & \vdots & \vdots \\ b_{m,1} & b_{m,2} & \dots & b_{m,n} & c_m \end{array}\right]}_{}$$

$m \times (n+1)$ matrix A, entries a_{ij}

i = row, j = column.

Typo last day: G.E. algorithm:

"until $i = \underline{m}$ or $j = \underline{n}$"

Last day: $GE \rightarrow$

$$\left[\begin{array}{ccc|c} \boxed{x} & x & x & x \\ 0 & \boxed{x} & x & x \\ 0 & 0 & \boxed{x} & x \\ 0 & 0 & 0 & X \end{array}\right]$$

— same solns as orig. lin sys.

echelon form.

X = nonzero
$\Rightarrow$ no sol'ns

lin sys has some sol'n.

Ex $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-1} & -2 & 1 \end{array}\right]$ echelon form.

$(2,:) \leftarrow (2,:) - (1,:)$

arbitrary

$-x_2 - 2x_3 = 1$ let $x_3 = t$

$\hookrightarrow x_2 = -1 - 2t$

$x_1 + 2(-1-2t) + 3(t) = 0$

$x_1 = 2 + t$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

(a line!)

not unique, family of sol'ns, one for each $t \in \mathbb{R}$

Ex $\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 4 \end{array}\right] \sim \cdots \sim$

swap row 1 and row 2

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

echelon form

row of all zeros

back sub

let $x_3 = t$

$x_2 = -5 - 4(t)$

$x_1 = -2 - 3(-5 - 4t) - 5(t) = 13 + 7t$

$\langle x_1, x_2, x_3 \rangle = \langle 13, -5, 0 \rangle + t\langle 7, -4, 1 \rangle$

Pivots via an example.

Eg $\left[\begin{array}{cccc|c} 1 & 5 & 0 & 1 & 2 \\ 10 & 50 & x & x & x \end{array}\right]$

Ex $\left[\begin{array}{cccc|c} \boxed{1} & 5 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 1 \end{array}\right]$ already in RREF

pivot columns

and x_1 and x_3 are pivots

free $\rightarrow x_4 = s$

pivot $\rightarrow x_3 = 1 - 2s$

free $\rightarrow x_2 = t$

pivot $\rightarrow x_1 = 2 - 5t - s$

- Leading entry variables determined by equs
- non-leading entry vars become free parameters

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Def'n: rank of a matrix is the number of pivot columns

(also equal to the # of nonzero rows ~~in~~ of the ~~coefficient~~ matrix in echelon form).

Notation: r = rank of the coefficient matrix
r_A = rank of the augmented matrix.

Thm ① if $r_A > r$ there are no sol'ns.

eg. $$\left[\begin{array}{ccccc|c} \boxed{x} & x & x & x & x & x \\ & & \boxed{x} & x & x & x \\ & & & & \boxed{x} & x \\ & & & & & x \end{array}\right] \qquad n=5$$

$r=3$

$r_A=4$

② otherwise, if $r = n$ (# of unknowns).
$(r_A = r)$

↳ unique sol'n, no free param.

else $r < n$, there are infinitely many sol'ns, and there are $n-r$ free parameters.

Ex a) $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $n=2$, $r=1$, $r_A=2$ no sol'ns

b) $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$ $n=2$, $r=2=r_A$ <u>Unique</u>

c) $\left[\begin{array}{ccc|c} \boxed{1} & 5 & 9 & 10 \\ 0 & \boxed{2} & 5 & 3 \\ 0 & 0 & \boxed{2} & 0 \end{array}\right]$ $r=3$, $r_A=3$, $n=3$ <u>Unique</u>

d) $\left[\begin{array}{ccc|c} 2 & 2 & 3 & 10 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $n=3$, $r=2=r_A$ non unique

$n-r=3-2=1$ free param

e) $\left[\begin{array}{cccc|c} 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $n=4$, $r=2=r_A$

zero

non-unique with 2 param family

Homogeneous linear systems

Recall in the non-unique answers case, we say $\vec{x} = \cancel{x_p + t} = \vec{q} + t\vec{a}_1 + s\vec{a}_2$ (example)

$\vec{x}$: solves sys for any s,t

$\vec{q}$: solves the system ($s=t=0$)

Q: what do $\vec{a}_1$ and $\vec{a}_2$ solve?

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= 0 \\ &\vdots \\ b_{m1}x_1 + b_{m2}x_2 + \dots b_{mn}x_n &= 0 \end{aligned}$$

zero RHS.

- homog. lin. sys. always (at least) one sol'n. $\vec{x} = \vec{0}$

exactly one sol'n when $r = n$

- if rank $r < n$ (# of variables) we have infinitely many sol'ns.

- if $\vec{x}$ is a sol'n of the homog. lin. sy. so is $t\vec{x}$

proof $b_{11}(tx_1) + b_{12}(tx_2) + \dots + b_{1n}(tx_n) = t\left[b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n\right] = t \cdot 0 = 0$

$\vdots$

- if $\vec{x}$ and $\vec{y}$ are sol'ns so is $\vec{x} + \vec{y}$

proof

$b_{11}(x_1 + y_1) + \dots + b_{1n}(x_n + y_n) = \overbrace{b_{11}x_1 + \dots + b_{1n}x_n}^{0} + \overbrace{b_{11}y_1 + \dots + b_{1n}y_n}^{0} = 0 + 0 = 0$

$\vdots$

(these are not true for RHS $\neq \vec{0}$)

Aside: computers, real numbers and floating point

computer storage / representation of #s in scientific notation

E.g. $3.67e{-16} = 3.67 \times 10^{-16}$

between 1 and 2 there are infinitley many real #s but 2^{53} ~~numbers~~ floating point numbers on the computer.

then ~~the~~ computer stores other numbers as

$$\underbrace{x}_{[1,2)} \times \underbrace{2^{b}}_{\text{power of } 2}$$

$[1,2)$: $2^{53} \simeq 9 \times 10^{15}$ #s.

Every real number gets approximated by one of these "floating point numbers".
The are so many floating point numbers so usually this doesn't matter... except when it does!
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