Math 152 lecture 8 2018-01-30

Last day: homogeneous linear systems

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ b_{m1}x_1 + b_{m2}x_2 + \dots + b_{mn}x_n &= 0 \end{aligned}$$

versus the general problem with data vector $\vec{c}$

replace data w/ zero.

Suppose we have the general linear system, if $\vec{x}_p$ is any particular sol'n and suppose $\vec{x}_h$ is the sol'n of the associated homog lin sys then

$$\vec{x} = \vec{x}_p + t\vec{x}_h \quad + s\vec{x}_{h_2} + \dots$$

depending on $n-r$

is the general sol'n. (to the original nonhomog problem).

param. formula for a line.

any sol'n of the nonhomog. prob.

solves the assoc. homog. prob.

also part of the homog. sol'n.

Note: if you subtract two sol'ns of the nonhomog. problem, the difference solves the assoc. homog. prob.

Ex
$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} \boxed{1} & 5 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 1 \end{array}\right]$$

① find any sol'n.

$x_3 + 2\cancel{x_4} = 1$ ← 0 ← arbitrary

$x_3 = 1$

$x_1 + \cancel{5x_2} = 2$ ← 0

$x_1 = 2$

$\vec{x}_p = (2, 0, 1, 0)$

② associated homog:

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

free variables

$x_3 + 2x_4 = 0$ ($x_4 = s$)

$x_3 = -2s$

$x_1 + 5x_2 + s = 0$ ($x_2 = t$)

$x_1 = -5t - s$

$$\vec{x}_h = t\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

general sol'n:

$$\vec{x} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{t\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_h}$$

Note: if I change the RHS $\vec{c}$, I can keep $\vec{x}_h$ and find a new $\vec{x}_p$

Eg: Same but let's make different $\vec{x}_p$

$x_3 + 2x_4 = 1$ choose $x_3 = 42 \Rightarrow x_4 = -41/2$

$\cancel{x_1} + 5x_2 - 41/2 = 2 \Rightarrow x_2 = \frac{2}{5}$

arbitrary, lazy → $41/2$

$\vec{x} = (41/2, 2/5, 42, -41/2) + u(-5, 1, 0, 0) + v(-1, 0, -2, 1)$

Linear Indep. (again) §3.4 in notes

Recall the def'n: vectors $\vec{x}_1 \in \mathbb{R}^m, \ldots, \vec{x}_n$ are linearly indep if and only if

sys of eqns, unknowns $s_1, \ldots, s_n$

$$s_1 \vec{x}_1 + s_2 \vec{x}_2 + \ldots + s_n \vec{x}_n = \vec{0}$$

has the unique sol'n $s_1 = s_2 = \ldots = s_n = 0$

Connection to a homog. lin sys. w/ Augmented matrix:

$m \times (n+1)$ matrix

$$\text{m rows} \rightarrow \begin{array}{c} \left[\begin{array}{cccc|c} | & | & & | & 0 \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n & \vdots \\ | & | & & | & 0 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \uparrow \\ s_1 & s_2 & \cdots & s_n & \text{RHS} \end{array} \end{array}$$

here the vectors form the columns of the matrix.

Linear indep test: put n vectors as columns of a matrix A (not augmented). If $\operatorname{rank}(A) = n$ then the vectors are linearly indep.

any n, any m

Ex are $\vec{x_1} = (1, 0, 5, 0)$, $\vec{x_2} = (1, 1, 1, 0)$ and $\vec{x_3} = (0, 1, -4, 1)$ linearly indep?

~~①~~ Can't use det b/c its not $\mathbb{R}^3$

② $s_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \vec{0}$

defn of l.i. gives this homog. lin. sys.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 5 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

~~rref(A)~~ echelon form

rank = 3 (# of nonzero rows)

and n = 3

So $s_3 = 0$

$s_2 + s_3 = 0 \Rightarrow s_2 = 0$

$s_1 = 0$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Unique

$s_1 = s_2 = s_3 = 0$

Ex Is $\vec{a} = (1,1,2)$ in $\text{span}\{\vec{x}_1, \vec{x}_2\}$
where $\vec{x}_1 = (-1,2,3)$ and $\vec{x}_2 = (5,0,1)$

i.e. $s_1 \vec{x}_1 + s_2 \vec{x}_2 = \vec{a}$

Can I find s_1, s_2?

$$s_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

3 eqns in 2 unknowns

nonhomog.

$$\left[\begin{array}{cc|c} -1 & 5 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \leftarrow 0=1$$

so no sol'ns

So $\vec{a} \notin \text{span}\{\vec{x}_1, \vec{x}_2\}$

Ex Are $\vec{a}, \vec{x}_1, \vec{x}_2$ l.i.?

$$s_1 \vec{x}_1 + s_2 \vec{x}_2 + s_3 \vec{a} = 0$$

$$\begin{bmatrix} -1 & 5 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unique sol'n
$s_1 = s_2 = s_3 = 0$

So yes they are l.i.

__Algorithm/Recipe__ ① to write $\vec{a}$ as a linear comb of $\{\vec{x}_1, \dots, \vec{x}_n\}$:

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n & \vec{a} \\ | & | & & | & | \end{array}\right]$$

$\vec{x}_i$ as columns

augment w/ $\vec{a}$

② do GE., soln is n-component vector $\vec{s}$

where

$$\vec{a} = s_1\vec{x}_1 + s_2\vec{x}_2 + \dots + s_n\vec{x}_n$$

if no soln, then $\vec{a}$ not in the span $\{\vec{x}_1, \dots, \vec{x}_n\}$.

__Aside__ computer arithmetic with floating point.. (cont)

key point: every arithmetic operation makes a small __relative error__ of about 10^{-16}

For us, we often see 10^{-16} or 10^{-15} after GE..

Eg. random $\vec{a}$, random $\vec{b}$, $\vec{c} := \vec{a} + \vec{b} + \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}$

lin indep? $A = \begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix}$

linearly dep if $\varepsilon = 0$, what if ε small

How do I know GE will not significantly increase this small ε?

$\rightarrow$ in principle ~~yes~~ it could!

Jim Wilkinson: "anyone unlucky enough to find such a matrix in practice has already been hit by a bus!"