Math 152 Lecture 9 2018-02-01

Example application: Resistor networks

Physics: Current I: charge flow (coulombs/sec, amps)
↓
C, measure of charge.

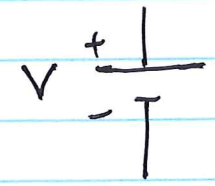
Voltage V: electrical potential (J/C, or volts)

So IV is power (J/sec or watts)

Some components in a circuit

① Resistors 

Obey's Ohm's law
current through it is proportional to the voltage drop across it
$V \sim I$ so $\boxed{V = IR}$
↑ resistance parameter
Ω = volts/amp

battery
↓
② Voltage source 

voltage increases by V from $-$ to $+$.

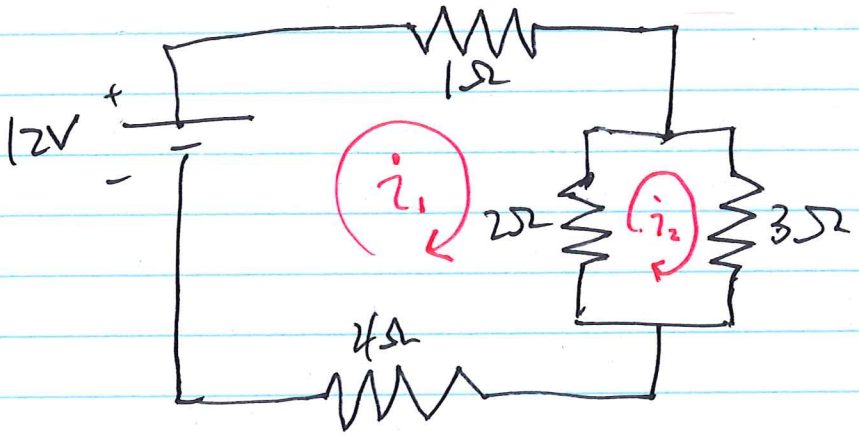
③ Current source 

current forced to be I (in direction of arrow)

↓
nontrivial to build one.

We build circuit/network by connecting these components/nodes with conductors/wires/branches

Ex



Assume: - network is planar (~~not~~ no crossing wires)

- "static", no change in time.
  ↳ transcient case in Ch 6.

Physics

↓

Math model

("loop currents"

Linear system

↓

Solve

{ Ohm's law ✓
Kirchoff's 1st law: Sum of the voltage drops around closed loops must be zero.
" 2nd ": Sum of the currents entering a node must be zero

Sign convention: current leaving a node is taken as negative

Main problem
- find current through the resistors and the voltage sources
- find the voltage drop across the current sources

sub problem ↓

Fundamental Network Problem – find the current through each voltage source and the voltage across each current sources.

Loop currents – math model

unknowns: loop currents $i_1, \dots, i_m$ in every "elementary loop" of the network ← cannot be divided.

$N = m + n$ unknowns

voltage drops $V_1, \dots, V_n$ across current sources

Equations (m): sum of voltage drops around each elementary loop are zero — Kirchoff 1st

(n): loop current matches the current source

Note: 2nd K. law satisfied automatically.

careful signs

Ex: as above. Unknowns: i_1 and i_2

Sum voltage drops over loop 1:

$$1i_1 + 2(i_1 - i_2) + 4i_1 - 12 = 0$$

$V = RI$ for 1Ω resistor

add current from multiple loops

crossed voltage source from − to + with voltage increase of 12 so voltage drop of −12

Sum voltage drops over loop 2:

$$3i_2 + 2(i_2 - i_1) = 0$$

current in direction of the loop.

So $\boxed{\begin{aligned} 7i_1 - 2i_2 &= 12 \\ -2i_1 + 5i_2 &= 0 \end{aligned}}$ → solve $\left[\begin{array}{cc|c} 7 & -2 & 12 \\ -2 & 5 & 0 \end{array}\right]$

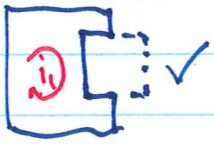
$\sim \cdots \sim$ (RREF) $\left[\begin{array}{cc|c} 1 & 0 & 60/31 \\ 0 & 1 & 24/31 \end{array}\right]$

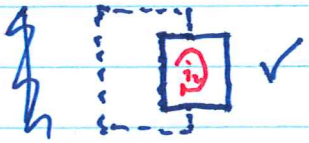
(of lin sys)

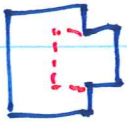
Soln is $i_1 = 60/31$, $i_2 = 24/31$

Ex: Find the current through the 2Ω resistor

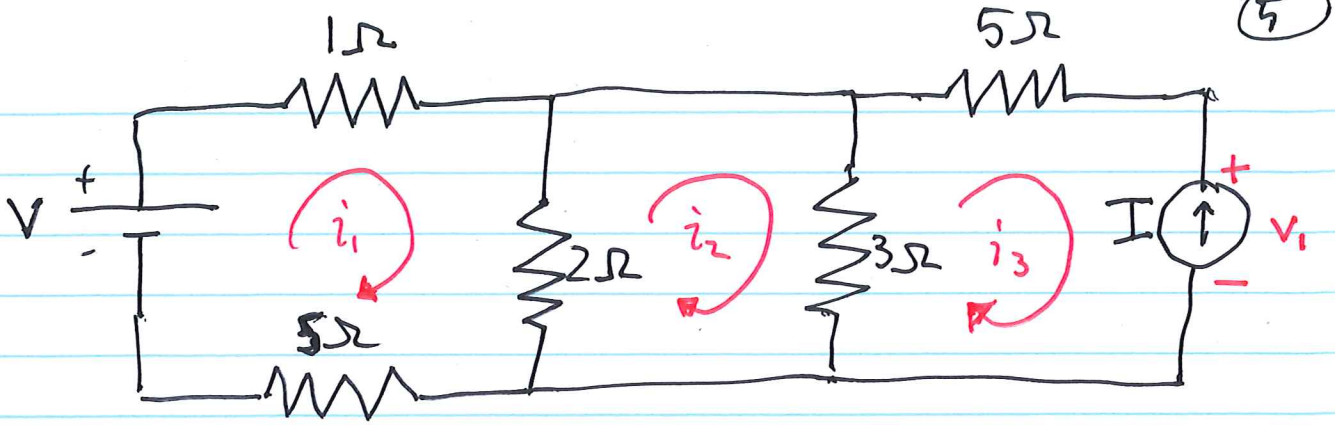
$i_1 - i_2 = \frac{60-24}{31} = \frac{36}{31}$ (downward)

loop 1:  ✓ Question from class

loop 2:  ✓

not an elementary loop:  ×

Ex



Note: V, I are parameters.

(a) Write down a linear system using loop currents

(b) solve linear sys when V=9 and I=1.

(c) solve "Fundamental Network Problem" ~~in terms (symbolically)~~ (symbolically) in terms of V and I

(a) Unknowns: i_1, i_2, i_3, V_1 ← 4 unknowns

If eqns

$$1i_1 + 2(i_1 - i_2) + 5i_1 = V$$

$$2(i_2 - i_1) + 3(i_2 - i_3) = 0$$

$$3(i_3 - i_2) + 5i_3 + V_1 = 0$$

$$i_3 = -I$$

voltage increase of V_1 from − to +

match current source with two loop.

$$\begin{aligned} 8i_1 - 2i_2 &= V \\ -2i_1 + 5i_2 - 3i_3 &= 0 \\ -3i_2 + 8i_3 + V_1 &= 0 \\ i_3 &= -I \end{aligned}$$

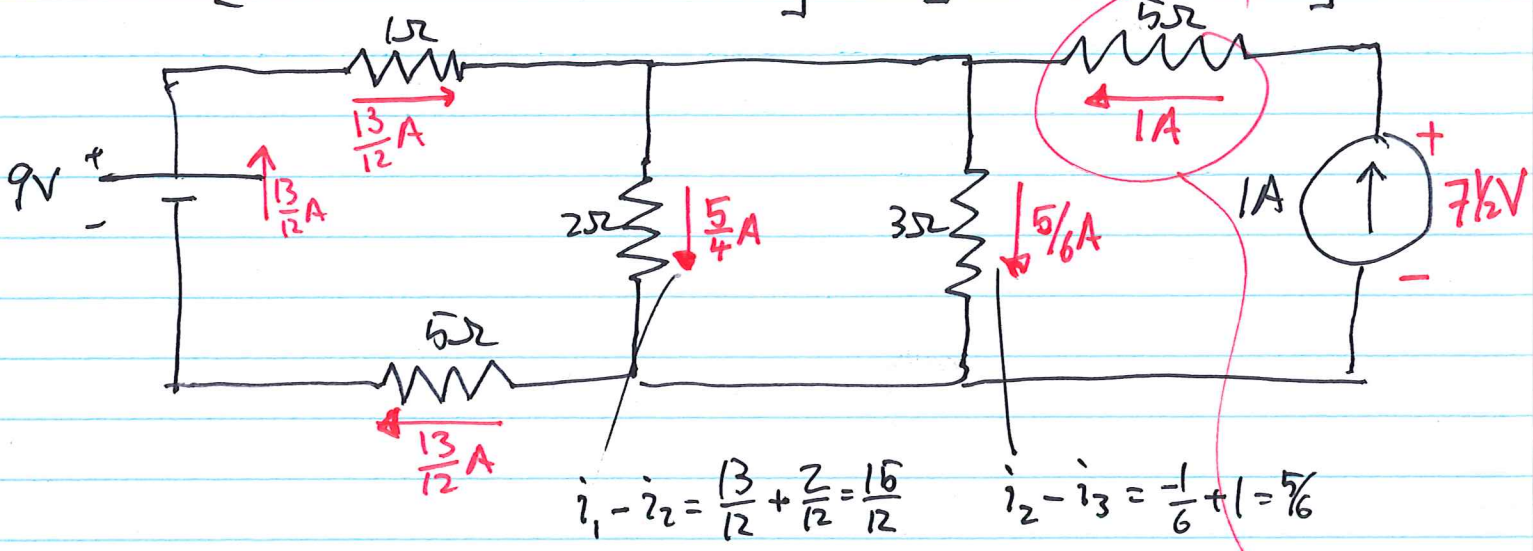
we want voltage drop so $-V_1$

... and we want voltage drop from + to − so $-V_1 = +V_1$

b)

$$\begin{matrix} i_1 & i_2 & i_3 & V_1 & \\ \end{matrix}$$

$$\left[\begin{array}{cccc|c} 8 & -2 & 0 & 0 & 9 \\ -2 & 5 & -3 & 0 & 0 \\ 0 & -3 & 8 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 13/12 \\ 0 & 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 15/2 \end{array}\right]$$



$i_1 - i_2 = \frac{13}{12} + \frac{2}{12} = \frac{15}{12}$

$i_2 - i_3 = \frac{-1}{6} + 1 = 5/6$

node direction here.

c)

$$\left[\begin{array}{cccc|c} 8 & -2 & 0 & 0 & V \\ -2 & 5 & -3 & 0 & 0 \\ \textcircled{3} & -3 & 8 & 1 & 0 \\ \textcircled{2} & \textcircled{3} & 1 & 0 & -I \end{array}\right] \begin{matrix} \\ \\ \\ \textcircled{2} \leftarrow \textcircled{2} + \frac{1}{4}\textcircled{1} \end{matrix} \sim \left[\begin{array}{cccc|c} \multicolumn{5}{c}{\text{cuchens}} \\ 0 & 9/2 & -3 & 0 & V/4 \\ & & & & \\ & & & & \end{array}\right]$$

$$\begin{matrix} \cancel{\textcircled{3} + \frac{2}{3}\textcircled{2}} \\ \sim \\ \textcircled{3} + \frac{2}{3}\textcircled{2} \end{matrix} \left[\begin{array}{cccc|c} & & & & \\ & & & & \\ 0 & 0 & 6 & 1 & V/6 \end{array}\right] \sim \begin{matrix} \\ \\ \textcircled{4} - \frac{1}{6}\textcircled{3} \end{matrix} \left[\begin{array}{cccc|c} 8 & -2 & 0 & 0 & V \\ 0 & 9/2 & -3 & 0 & V/4 \\ 0 & 0 & 6 & 1 & V/6 \\ 0 & 0 & 0 & -1/6 & -I - V/36 \end{array}\right]$$

$$\sim \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & 1 & +6I + V/6 \end{bmatrix} \quad -6\textcircled{4}$$

$$\sim \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 6 & 0 & -6I \end{bmatrix} \quad \textcircled{3}-\textcircled{4}$$

$$\sim \begin{bmatrix} & & & & \\ & & & & \\ 0 & 0 & 1 & 0 & -I \\ & & & & \end{bmatrix} \quad \tfrac{1}{6}\textcircled{3}$$

$$\sim \begin{bmatrix} & & & & \\ 0 & \frac{9}{2} & 0 & 0 & V/4 - 3I \\ & & & & \end{bmatrix} \quad \textcircled{2}+3\textcircled{3}$$

$$\sim \begin{bmatrix} & & & & \\ 0 & 1 & 0 & 0 & V/18 - \frac{2}{3}I \\ & & & & \end{bmatrix} \quad \tfrac{2}{9}\textcircled{2}$$

$$\sim \begin{bmatrix} 8 & 0 & 0 & 0 & V + \frac{1}{9}V - \frac{4}{3}I \\ & & & & \\ & & & & \end{bmatrix} \quad \textcircled{1}+2\textcircled{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{10}{72}V - \frac{4}{24}I \\ 0 & 1 & 0 & 0 & \frac{1}{18}V - \frac{2}{3}I \\ 0 & 0 & 1 & 0 & -I \\ 0 & 0 & 0 & 1 & \frac{1}{6}V + 6I \end{bmatrix} \quad \tfrac{1}{8}\textcircled{1}$$

RREF

c) $i_1 = \frac{5}{36}V - \frac{1}{6}I$ and $V_1 = \frac{1}{6}V + 6I$

i_1: current upward through battery

V_1: voltage jump in direction of specified current.