Math 152, Lecture 12

~~then~~ output $\quad$ input

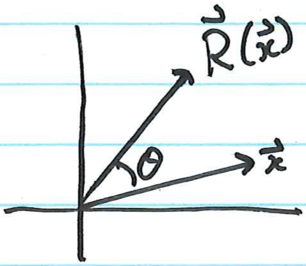
$\vec{y} = \vec{T}(\vec{x}) \qquad \vec{T} : \mathbb{R}^n \to \mathbb{R}^m$

last day: Linear transforms §4.2

$\vec{T}(s\vec{x} + \vec{z}) = s\vec{T}(\vec{x}) + \vec{T}(\vec{z})$

Rotation in 2D

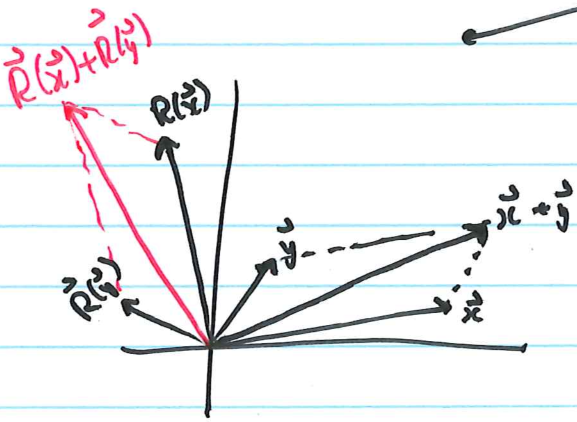
— rotation by θ radians ccw (counter clockwise)

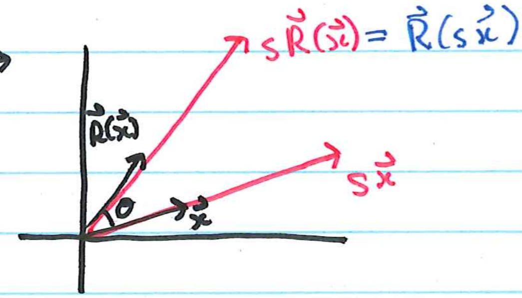


— this is a linear transform:

need: (i) $\vec{R}(\vec{x}+\vec{y}) = \vec{R}(\vec{x}) + \vec{R}(\vec{y})$

(ii) $\vec{R}(s\vec{x}) = s\vec{R}(\vec{x})$

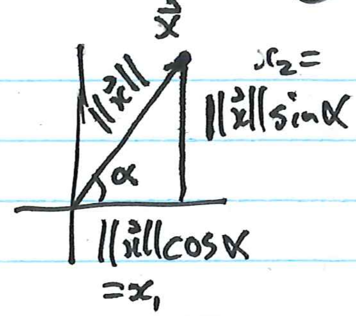




Alternatively: construct a 2×2 matrix Rot_θ so that

$$\vec{R}(\vec{x}) = Rot_\theta\,\vec{x} = \boxed{\begin{matrix} x & x \\ x & x \end{matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(we will see that all linear transforms can be written as matrix multiplication.)



Construction: Using polar coors
$(x_1, x_2) \to \{$ length and angle.

Write: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{x}\| \cos\alpha \\ \|\vec{x}\| \sin\alpha \end{bmatrix}$ ← angle α

"easy" to apply the rotation:

$$\vec{y} = \cancel{Rot_\theta \vec{x}} = \vec{R}(\vec{x}) = \begin{bmatrix} \|\vec{x}\| \cos(\alpha+\theta) \\ \|\vec{x}\| \sin(\alpha+\theta) \end{bmatrix}$$

Trig id

$$\vec{y} = \|\vec{x}\| \begin{bmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \cos\alpha\sin\theta + \sin\alpha\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \|\vec{x}\| \cos\alpha \\ \|\vec{x}\| \sin\alpha \end{bmatrix}$$ check

$$= Rot_\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Rot_\theta \vec{x}$$

where $\boxed{Rot_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$ ∎

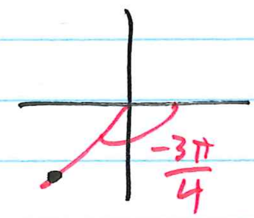
Aside

$\tan\alpha = \dfrac{x_2}{x_1}$

$\alpha = \arctan\left(\dfrac{x_2}{x_1}\right)$

On computer: $\text{atan}(x_2/x_1)$

Eg say $x_1 = -1$, $x_2 = -1$



$\text{atan}(-1/-1) = \dfrac{\pi}{4}$

WTF?

correct is $-\dfrac{3\pi}{4}$ or $\dfrac{5\pi}{4}$

Soln: use "atan2"
$\text{atan2}(x_2, x_1)$

So rotation of a vector by angle θ is equivalent to mult. by the 2×2 matrix Rot_θ.

Eg $\quad Rot_{\pi/4} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$Rot_{\pi/2} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \rightarrow$ Eg $Rot_{\pi/2} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix}$

Familar eh?! $\vec{a}^{\perp}$ from ~~lecture~~ way back.

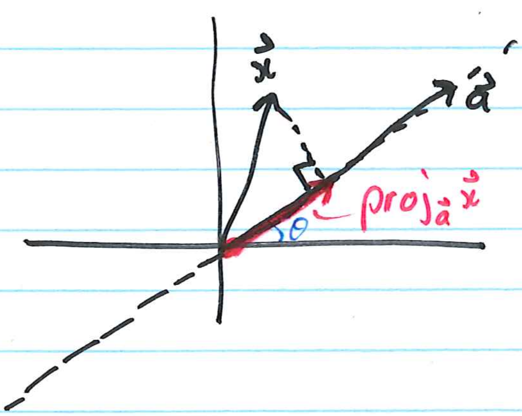
$Rot_{\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$Rot_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ identity matrix from last day. $I\vec{x} = \vec{x}$

↳ corresponding identity transform: $f(\vec{x}) = \vec{x}$

Projection in 2D

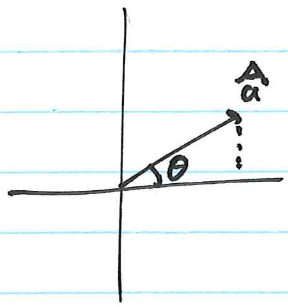
Recall: $proj_{\vec{a}} \vec{x} = \left(\frac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|^2} \right) \vec{a}$



We will show this is a linear transform by finding an equivalent matrix, given $\vec{a}$.

↳ ~~$Proj_{\vec{a}}$~~, 2×2 matrix $Proj_{\theta}$ where θ is the angle $\vec{a}$ makes with x_1 axis.

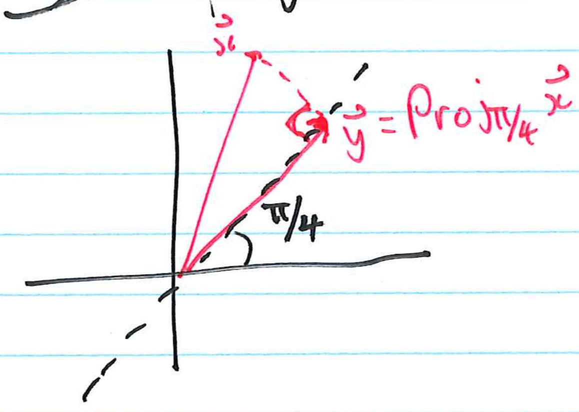
Construction: Suppose $\hat{a}$ is a unit vector
$$\hat{a} = (\cos\theta, \sin\theta)$$



$$\vec{y} = \text{proj}_{\hat{a}}\,\vec{x} = (\vec{x}\cdot\hat{a})\,\hat{a}$$
$$= (x_1\cos\theta + x_2\sin\theta)\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$$
$$= \begin{bmatrix} x_1\cos^2\theta + x_2\sin\theta\cos\theta \\ x_1\sin\theta\cos\theta + x_2\sin^2\theta \end{bmatrix}$$
$$= \boxed{\underbrace{\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}}_{\cancel{\text{Proj}_{\hat{a}}}\ \text{Proj}_\theta}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Trig id gives alternative: $\text{Proj}_\theta = \frac{1}{2}\begin{bmatrix} 1+\cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1-\cos 2\theta \end{bmatrix}$

Eg projection onto the line $x_2 = x_1$ "collapses" all points onto the line.



$$\cos\pi/4 = \frac{1}{\sqrt{2}} = \sin\pi/4$$

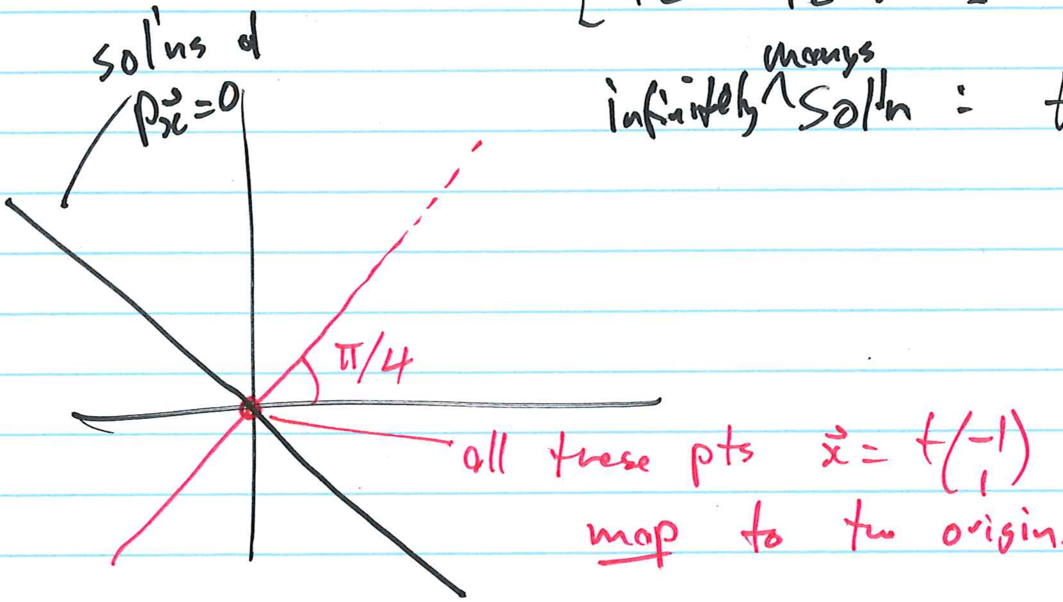
$$\text{Proj}_{\pi/4} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

<u>Note</u> $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ has rank 1

↳ OK... solve $P\vec{x} = \vec{0}$

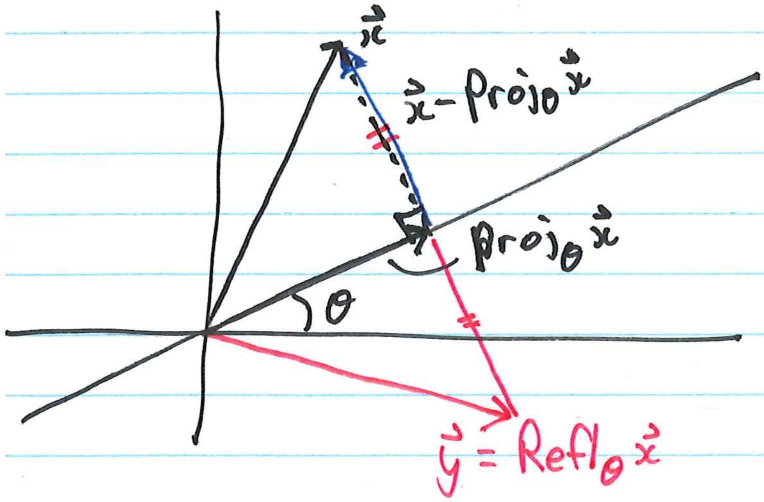
$$\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

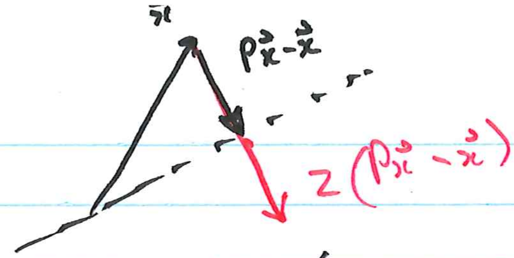
infinitely many soln : $t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{x}$



<u>Note</u>: rotations have rank 2, $R\vec{x} = \vec{0}$ has unique soln : $\vec{x} = \vec{0}$

Reflections in 2D





$$\vec{y} = \vec{x} + 2\left(\text{Proj}_\theta \vec{x} - \vec{x}\right)$$

$$= 2\,\text{Proj}_\theta \vec{x} - \vec{x}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2\,\text{Proj}_\theta \vec{x} - I\vec{x}$$

$$= \underbrace{\left(2\,\text{Proj}_\theta - I\right)}_{\text{Refl}_\theta} \vec{x}$$

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so $\text{Refl}_\theta = \left(2\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

alternatively : $\text{Refl}_\theta = \begin{bmatrix} 1+\cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1-\cos 2\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\boxed{\text{Refl}_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}$$

Eg $\text{Refl}_{\pi/4} = \begin{bmatrix} \cos \pi/2 & \sin \pi/2 \\ \sin \pi/2 & -\cos \pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So reflection through $x_2 = x_1$ exchanges the coordinates of $\vec{x}$

~~Th~~ **Theorem** any linear transform $T: \mathbb{R}^n \to \mathbb{R}^m$
is equivalent to multiplication by a $m \times n$ matrix $\mathbb{T}$.

We can say $\mathbb{T}$ is the matrix representation of the transform.

Proof is constructive: we can find it explicitly.