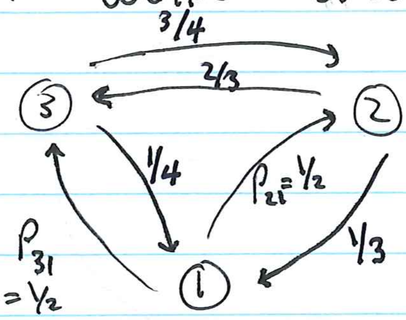
Math 152 Lecture ~~14~~

Recall: random walks: around a "state space":

Ex



$$\mathbb{P} = \begin{bmatrix} 0 & 1/3 & 1/4 \\ 1/2 & 0 & 3/4 \\ 1/2 & 2/3 & 0 \end{bmatrix}$$

transition probabilities

$x_i^{(n)}$: probability of being in state i at time n

$\vec{x}^{(n)}$: vector of state probabilities

P_{ij}: probability of leaving state j going to state i

$$\underline{\text{Model}}: \quad \vec{x}^{(n+1)} = \mathbb{P}\,\vec{x}^{(n)}$$

Eg. $\vec{x}^{(1)} = \mathbb{P} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$, $\quad \vec{x}^{(2)} = \mathbb{P}\vec{x}^{(1)} = \mathbb{P}(\mathbb{P}\vec{x}^{(0)})$

$$= \mathbb{P}^2 \vec{x}^{(0)} \approx \begin{bmatrix} 0.2917 \\ 0.375 \\ 0.3333 \end{bmatrix}$$

Example from last day

(c) What is the probability of being in state 2 after a <u>long time</u>

$$\lim_{n\to\infty} \vec{x}^{(n)} = \lim_{n\to\infty} \mathbb{P}^n \vec{x}^{(0)}$$

For now, numerical evidence... (math later)

(~~matlab~~ matlab / octave)

$$\vec{x}^{(n)} \to \begin{bmatrix} 0.2264 \\ 0.3962 \\ 0.3774 \end{bmatrix}$$

(d) How ~~th~~ much does the "initial condition" $\vec{x}^{(0)}$ matter? (experimentally)

$\lim_{n\to\infty} x_2^{(n)}$ <u>seems</u> to be 0.3962 for many different $\vec{x}^{(0)}$.

"stable process"

Example
notes.pdf : "specific but somewhat nerdy example" 4.1

Ydnew <u>vs</u> Xavier

Ydnew: spell works 1/2 the time.

Xavier: goes first; spell works 1/3 of the time.

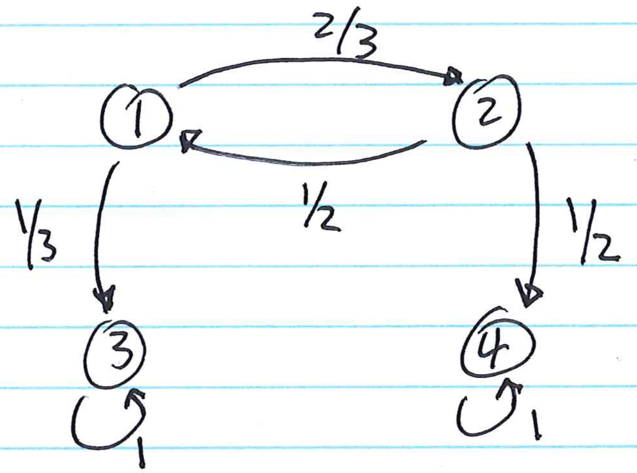
} see notes.pdf

long duel: play until someone wins (other knocked out)
(experimentally, each wins half the time).

short duel: Xavier as 3 attempts. afterwhich Ydnew wins by default.

(a) describe duels as random walks.

4 possible <u>states</u>:
1. Xavier's turn (no winner)
2. Ydnew's turn (" ")
3. X has won
4. Y has won.



$$\mathbb{P} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 2/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}$$

b) analyze probability that Y will win the short duel. (X has 3 attempts: XYXYX)

$$\vec{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x}^{(5)} = \mathbb{P}^5 \vec{x}^{(0)} \approx \begin{bmatrix} 0 \\ 7.4\% \\ 48.1\% \\ 44.4\% \end{bmatrix}$$

matlab

Y wins 52% of the time

c) long duel: matlab experiments confirm

$$\lim_{n\to\infty} \mathbb{P}^n \vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

experiments

Ex What if X and Y have a shield which blocks one ~~succef~~ successful spell ~~caste~~. cast.

States:

1. no winner, Xavier's turn, X shield, Y shield
2. " " X no shield, "
3. " " " , Y no shield
4. " " X shield, " .

5–8. repeat but Ydnew's turn

9. Xavier has won.
10. Ydnew " " .

X spell breaks Y shield

X can't win on 1st round

$$
\mathbb{P} = \left[\begin{array}{cccc|cccc|cc}
0 & 0 & 0 & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac12 & \frac12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac12 & \frac12 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac12 & 0 & 0 \\
\hline
\frac23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac13 & \frac23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac13 & 0 & 0 & \frac23 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \frac13 & \frac13 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac12 & \frac12 & 0 & 0 & 1
\end{array}\right]
$$

... matlab experiment suggests Y wins the short duel 77.8% of the time. (Xavier ~~lost or~~ didn't win)

... $\lim_{n\to\infty}$ seems to have Y winning 62.5% of the time.

Transpose

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

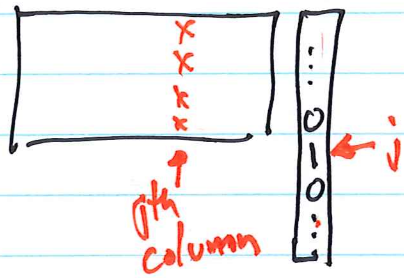
def'n: if A is an $m \times n$ matrix entries a_{ij} then A^T (the transpose of A) is an $n \times m$ matrix with entries $a^T_{ij} = a_{ji}$.

- columns of A become rows of A^T.
  rows " " " columns " ".
- $(A^T)^T = A$ for every matrix
- save vertical space when writing
  $$[1 \;\; 2 \;\; 3 \;\; 4]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$
- If A is square $(n \times n)$ and $A^T = A$ we say A is symmetric.
- If $\vec{x}$ and $\vec{y}$ are column vectors in $\mathbb{R}^n$ then
  $$\vec{x}^T \vec{y} = \quad = \vec{x} \cdot \vec{y} = \text{scalar}$$
- $(AB)^T = B^T A^T$ (order switches).

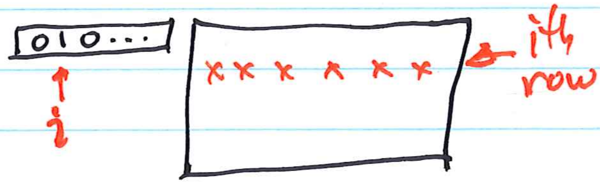
vector of all zeros, but jth ~~position~~ position is 1.

Operations on $\vec{e}_j$ vectors

Say A is $m \times n$ matrix.

① $A\vec{e}_j =$ 

$=$ jth column of A

② $\vec{e}_i^T A =$ 

$=$ ith row of A

③ $\vec{e}_i^T A \vec{e}_j = a_{ij}$

Note: transpose requires no arithmetic

↳ computer can even use the same memory

↳ cheap to do an ~~inverse~~. transpose.

contrast that fact with...

Matrix Inverse

Square $n \times n$ matrix A. Suppose A has rank n so $A\vec{x} = \vec{b}$ has a unique sol'n. $\vec{x}$

(recall → GE on Augmented matrix...)

told us rank — gave us $\vec{x}$

The operation of finding $\vec{x}$ from given $\vec{b}$ turns out to be a <u>linear</u> <u>transform</u>

Thus there is a matrix B such that

$$\vec{x} = \underset{\substack{\uparrow \\ \text{transform}}}{B}\,\vec{b} \qquad (*)$$

But $A\vec{x} = \vec{b}$ so multiply $(*)$ by A:

$$\underbrace{\cancel{A\vec{x}}}_{\vec{b}} = A B \vec{b}$$

Therefore $AB = \underset{\substack{\uparrow \\ \text{identity matrix.}}}{I}$ (because $\vec{b} = AB\vec{b}$)

We say B is the <u>inverse</u> <u>matrix</u> of A and we write $B = A^{-1}$.