Math 152 Lecture 15

Last day: matrix inverse $AA^{-1} = I = A^{-1}A$ (can also show)

- A: square $n \times n$ matrix, rank n (see this soon)
- A^{-1}: matrix inverse, $n \times n$
- I: $n \times n$ identity

How to find A^{-1}? today

If we have A^{-1}, we can solve

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

philosophy: finding A^{-1} must be at least as hard as GE...

Ex Show $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has inverse $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

check: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Solve $A\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$: $\vec{x} = A^{-1}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

Finding A^{-1}: Recall / we can find the matrix corresponding to a linear transform ($B = A^{-1}$) by calculating the action of the transform on special $\vec{e}_j$ vectors.

Specifically the columns of B are

$$\vec{b}_j = B\vec{e}_j$$

← vector all zeros, but for the jth entry which is 1.

But we know A

$$\vec{b}_j = A^{-1}\vec{e}_j$$

$$A\vec{b}_j = AA^{-1}\vec{e}_j$$

$$I\vec{e}_j$$

$$\boxed{A\vec{b}_j = \vec{e}_j}$$

← Solve this for $j = 1, \dots, n$

Can solve for all the $\vec{b}_j$ simultaneously by forming:

$$\left[A \,\vdots\, \vec{e}_1\ \vec{e}_2 \dots \vec{e}_n\right] = \left[A \,\vdots\, I\right]$$

$$\underset{\text{to rref}}{\overset{\text{GE}}{\sim}} \left[I \,\vdots\, A^{-1}\right]$$

Gives an <u>algorithm</u> to find A^{-1}.

Ex Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \underset{\substack{(2)-(1) \\ (3)-(1)}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \underset{(3)-3(2)}{} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right]$$

$$\underset{(3)/2}{\sim} \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

$$\underset{\substack{(1)-(3) \\ (2)-2(3)}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

$$\underset{(1)-(2)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

(the boxed right half is A^{-1})

need A to have rank 3, to get I here

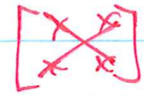
Note:
- If $n \times n$ A does not have rank n, then A is not invertible
- $(AB)^{-1} = B^{-1}A^{-1}$ note order

Determinants

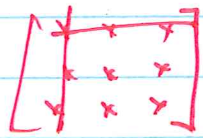
scalar value associated with a square $n \times n$ matrix.

Review:

A 2×2, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$



A 3×3, $\det(A) = a_{11}\det M_{11} - a_{12}\det M_{12}$



$+ a_{13}\det M_{13}$

$$= \sum_{j=1}^{3} (-1)^{j+1} a_{1j} \det M_{1j}$$

where M_{ij} is the $(n-1)\times(n-1)$ submatrix of A obtained by crossing off the i^{th} row and j^{th} column of A

In general, A $n\times n$ $\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j})$

expansion along first row

det of $(n-1)\times(n-1)$ matrix → use formula recursively

Ex calc the det of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

$$\det(A) = 1 \det\begin{pmatrix} 6 & 7 & 8 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{pmatrix} - 2 \det\begin{pmatrix} 5 & 7 & 8 \\ 9 & 1 & 2 \\ 3 & 5 & 6 \end{pmatrix}$$

$$+ 3 \det\begin{pmatrix} 5 & 6 & 8 \\ 9 & 0 & 2 \\ 3 & 4 & 6 \end{pmatrix} - 4 \det\begin{pmatrix} 5 & 6 & 7 \\ 9 & 0 & 1 \\ 3 & 4 & 5 \end{pmatrix}$$

$$= 1\left[6 \overset{-4}{\det\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}} - 7 \det(\quad) + 8 \det(\quad)\right]$$

$$-2\left[5 \det(\quad) - 7 \det(\quad) + 8 \det(\quad)\right]$$

$$+3\left[5 \det(\quad) - 6 \det(\quad) + 8 \det(\quad)\right]$$

$$-4\left[5 \det(\quad) - 6 \det(\quad) + 7 \det(\quad)\right]$$

$= \ldots = 0$ (not invertible)

Properties of the determinant: (w/o proof)

(1) $\det(A) \neq 0$ iff A is invertible.

(2) $\det(I_n) = 1$ (direct ~~com~~ calculation)

(3) $\det(AB) = \det(A)\det(B)$

Consequences: ① if A is not invertible. then $\det(A) = 0$ thus $\det(AB) = 0$ for any matrix B thus AB also not invertible.

② $\det(A^{-1}) = \dfrac{1}{\det(A)}$

(4) $\det(A) = \det(A^T)$

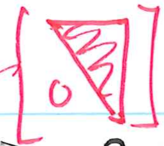
(5) You can use determinants to find (long) formulae for matrix inverses, called Cramer's Rules.

(6) Row operations on matrix, change the determinant in well-defined ways

$A \sim B$ row switch — $\det(B) = -\det(A)$

$A \sim B$ multiply a row by "a" — $\det(B) = a\det(A)$

$A \sim B$ add a multiple of a row — $\det(B) = \det(A)$



(7) det(diagonal matrix) = product of diagonal.

det(upper triangular) = " " "

det(lower ") = " " "

Eg: $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - 0\,a_{12}$

$= a_{11}a_{22}$ = product of diagonal

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} = \text{"}$$

We can use (6) and (7) to compute determinants of large matrices...

Eg $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 3 & 5 \\ 3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & \not{4}\;-2 & \not{6}\;-9 \end{bmatrix}$

Row ops: exchange 1 & 2
(3) − 3(1)
(3) + 2(2)

$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix} = B$

$\det(B) = (1)(1)(-1) = -1$
↑ upper tri

$\det(A) = -\det(B) = +1$

Ex $A = \begin{bmatrix} 1 & 2 & -2 & -7 \\ 1 & 2 & -1 & -5 \\ 0 & 3 & 0 & -3 \\ -1 & 4 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & -7 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} = B$

- no row mult.
- one row switch
- row + mult of other row

$\det(B) = (1)(3)(1)(2) = 6$

$\det(A) = -\det(B) = -6$

Ex If A $\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, what is $\det(A)$?

$\det(A) = 0$, A is not invertible

Trick for computing determinants:

Ex
$$A = \begin{bmatrix} 5 & 0 & 0 \\ 8 & 67 & 1 \\ \pi & 10 & 1 \end{bmatrix} \leftarrow \ddot{\smile}$$

$$\det(A) = 5 \det\begin{pmatrix} 67 & 1 \\ 10 & 1 \end{pmatrix} + 0 + 0 = 285$$

Ex
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

$\ddot{\smile}$

$\det(A) = \det(A^T)$ — easy

What if
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 1 & 6 & 2 \end{bmatrix} \leftarrow ?$$

$$\det(A) = -\det\left(\begin{bmatrix} 0 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & 6 & 2 \end{bmatrix}\right) = 5$$

But we can do this "trick" directly:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij}$$ expand across row i

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij}$$ expand across column j

Ex $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 1 & 6 & 2 \end{bmatrix}$

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ ← $(-1)^{i+j}$

$\det(A) = -0 \det\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$

$+ 5 \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$- 0 \det\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$

$= 5(2-1) = 5$

~~$= 5(10-12) = -10$~~