Math 152 Lecture 16

Last ~~day~~ week : inverses, determinants

Ex For what values of a is this matrix invertible?

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ a & 5 & 0 \\ 1 & 6 & 2 \end{bmatrix}$$

$$\det(A) = -a \det\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix} + 5 \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 0$$

$$= -a(2-6) + 5(2-1) = 4a + 5$$

Matrix not invertible when $a = -5/4$

Ex Consider the 1×1 matrix : x

when is x invertible? inverse is $x^{-1} = \frac{1}{x}$

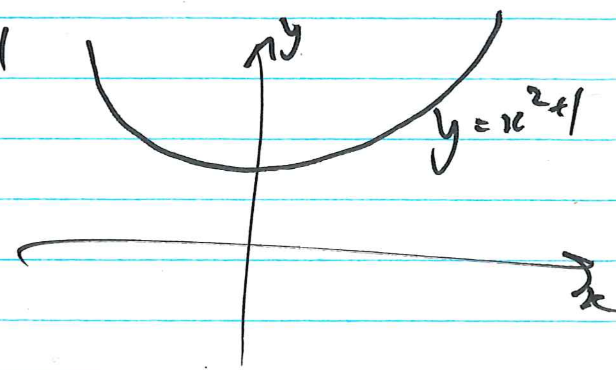
invertible when $x \neq 0$

$\det(x) = x$ (Mnemonic device to recall $\det \neq 0 \Leftrightarrow$ matrix invertible)

§5 Complex numbers

arise "naturally" when studying roots of polynomials

Ex $x^2 + 1 = 0 \Rightarrow x^2 = -1$



introduce "i" such that

$$\boxed{i^2 = -1}$$

With this $x^2 + 1 = 0$ is solved by $x = \pm i$

Ex $x^2 + x + 1 = 0$ quad. formula:

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2} = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2}$$

Complex number

$z = x + iy$, where $z \in \mathbb{C}$ ← set of complex numbers

$x, y \in \mathbb{R}$ ← set of real numbers

x: real part of z, $\mathrm{Re}(z)$
y: imaginary part of z, $\mathrm{Im}(z)$

note: $\mathrm{Im}(z)$ is <u>real</u>

Fundamental Theorem of Algebra : (FTA)

any polynomial of degree n can be completely factored in terms of n roots, possibly complex, possibly repeating.

poly: $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n$

(here $a_1, \ldots, a_n$ are given coefficients)

$$= (z - r_1)(z - r_2) \ldots (z - r_n).$$

Note: finding these roots ~~is~~ can be hard

↳ no general formula for $n > 4$.

In practice, we use computer approx of the roots, with iterative methods.

Complex number arithmetic

$z = x + iy$, $u = s + it$ $x, y, s, t \in \mathbb{R}$

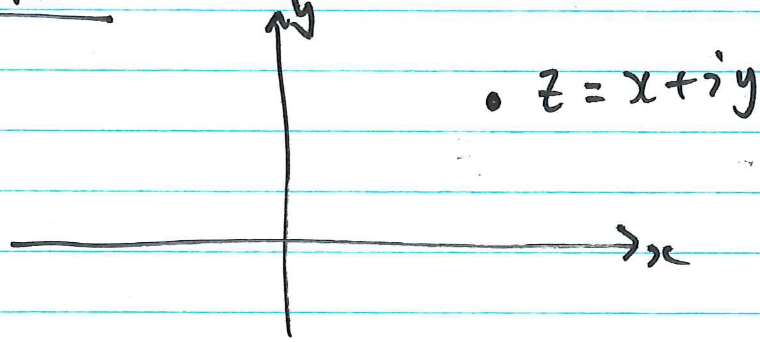
Add: $u + z = x + s + i(y + t)$

Mult. by real scalar $c \in \mathbb{R}$, $cz = cx + i(cy)$

$$\frac{z}{c} = \frac{x}{c} + i\frac{y}{c}$$

Can define length (a.k.a. modulus): $|z| = \sqrt{x^2+y^2}$

draw complex number as a point in the complex plane



So we can think of complex numbers as 2-dim vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and the arithmetic above follows.

But, we can multiply complex numbers

$$zu = (x+iy)(s+it)$$

$$= xs + xit + siy + \overset{-1}{i^2}yt$$

$$= \underbrace{(xs-yt)}_{\mathbb{R}} + i\,\underbrace{(xt+ys)}_{\mathbb{R}}$$

note: $i^2=-1$

memorize this

but not this.

Ex $\quad (1+i)(2-3i) = (2+\overset{-3i^2}{3}) + i(2-3) = 5-i.$

Complex conjugate

also denoted z^*

$z = x+iy$ then $\bar{z} = x-iy$

$\bar{z}$ is the complex conjugate of z,

"z bar" or "z conjugate"

Note: complex roots of polynomials with real coefficients come in conjugate pairs

Properties of conjugate

- $\dfrac{z+\bar{z}}{2} = \mathrm{Re}(z) = x$
- $\dfrac{z-\bar{z}}{2i} = y = \mathrm{Im}(z)$
- $\overline{zu} = \bar{z}\,\bar{u}$ (show by writing it out)
- $z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

Division of two complex numbers

$u = s+it, \quad z = x+iy \quad (z \neq 0)$

not both $x, y = 0$

Complex product

$$\frac{u}{z} = \frac{u\bar{z}}{z\bar{z}} = \frac{u\bar{z}}{x^2+y^2} = \frac{1}{x^2+y^2}\Big((s+it)(x-iy)\Big)$$

memorize this trick

scalar division

$$= \frac{1}{x^2+y^2}\Big((sx+ty)+i(tx-sy)\Big)$$

$$= \left(\frac{sx+ty}{x^2+y^2}\right) + i\left(\frac{tx-sy}{x^2+y^2}\right)$$

<u>Ex</u> $z = 1+i$, $u = 2-3i$, do complex arith:

- $z + u = 3 - 2i$
- $3z = 3 + 3i$
- $uz = (2+3) + (2-3)i = 5 - i$
- $\bar{z} = 1 - i$
- $|z|^2 = z\bar{z} = (1+i)(1-i) = 1 + 1 = 2$
- $\frac{u}{z} = \frac{u\bar{z}}{z\bar{z}} = \frac{1}{2}\Big((2-3i)(1-i)\Big) = \frac{1}{2}\Big(2-3+i(-2-3)\Big)$

  $= -\frac{1}{2} - \frac{5}{2}i$

Linear Algebra w/ complex numbers

Ex $$A = \begin{bmatrix} i & 0 & 3 \\ 1 & i & -1 \\ 0 & 1 & 3+i \end{bmatrix}$$

$$\det(A) = i \det\begin{pmatrix} i & -1 \\ 1 & 3+i \end{pmatrix} - 0 + 3 \det\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$= i\left(i(3+i) + 1 \right) + 3(1 - 0i)$$

$$= i\left(3i + \cancel{i^2 + 1} \right) + 3$$

$$= 3i^2 + 3 = 3(-1) + 3 = 0$$

Ex b/c $\det(A) = 0$, there are infinite sol'ns to $A\vec{x} = \vec{0}$ (homog. problem). Find ~~them~~ one nontrivial sol'n.

here $\frac{3}{i} = \frac{3i}{i^2} = \frac{3i}{-1}$

$$\left[\begin{array}{ccc|c} i & 0 & 3 & 0 \\ 1 & i & -1 & 0 \\ 0 & 1 & 3+i & 0 \end{array}\right] \times \tfrac{1}{i} \sim \left[\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ & \text{(same)} & & \\ & & & \end{array}\right]$$

$$\underset{(2)-(1)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & i & -1+3i & 0 \\ 0 & 1 & 3+i & 0 \end{array}\right]$$

$$\times -i \sim \left[\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & 1 & \cancel{-1+3i} & 0 \\ 0 & 1 & 3+i & 0 \end{array}\right]$$

$i + 3$

$$\underset{(3)-(2)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & 1 & 3+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

knew this would happen.

All sol'ns to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = (3i, -3-i, 1)\, t$$

any complex #

One sol'n is when ~~A~~ $t=1$, $\underbrace{\vec{x} = (3i, -3-i, 1)}$

3 scalar complex numbers makes up a 3-dimensional vector $\longrightarrow \vec{x} \in \mathbb{C}^3$

(c.f. a single complex number can be thought of as a 2-dim vector).

Complex exponential

e^z : complex exponential $e^z : \mathbb{C} \to \mathbb{C}$

- should match e^x whenever z is real
- " satisfy other properties of the exponential fcn.

e.g., $e^{z_1+z_2} = e^{z_1} e^{z_2}$ (2)

e.g. $\frac{d}{dt} e^{zt} = z e^{zt}$ $\quad t \in \mathbb{R}$

start w/ $Re(z) = 0$

$$e^{ib} = \cos b + i \sin b.$$

Euler's (take as given for now)

$$e^z = e^{\overbrace{a}^{\mathrm{Re}(z)} + i\overbrace{b}^{\mathrm{Im}(z)}} = e^a e^{ib} \overset{\text{by (2)}}{=} e^a(\cos b + i \sin b)$$

$$= (\underbrace{e^a \cos b}_{\mathbb{R}}) + i(\underbrace{e^a \sin b}_{\mathbb{R}}).$$

b is some kind of angle

... next day.