Math 152 Lecture 17

Last day: complex numbers $z = \underset{\mathbb{R}}{a} + \underset{i^2=-1}{i}\,\underset{\mathbb{R}}{b}$

complex arithmetic ...

complex exponential:

$$e^z = e^{a+ib}$$
$$= e^a e^{ib}$$
$$= e^a \underbrace{(\cos b + i \sin b)}_{\text{Euler's Formula}}$$

complex

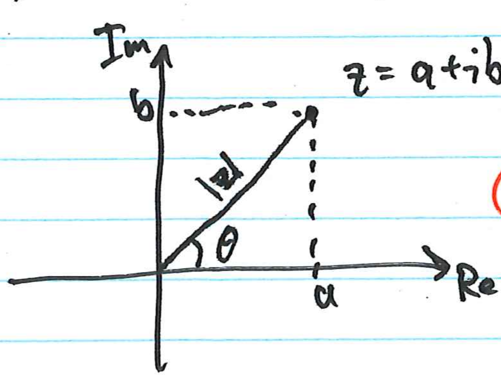
scalar, time

__Ex__

$$e^{zt} = e^{(a+ib)t} = e^{at+ibt}$$
$$= \underset{\text{growth/decay}}{e^{at}} \big(\underbrace{\cos(bt) + i\sin(bt)}_{\text{oscillation}}\big)$$

↳ important in solving differential equations.

Polar form of complex number



$z = a + ib$

modulus/length/magnitude

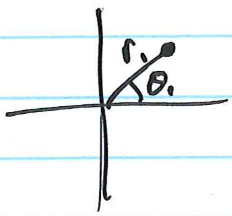
$$a = |z| \cos\theta$$
$$b = |z| \sin\theta$$

argument

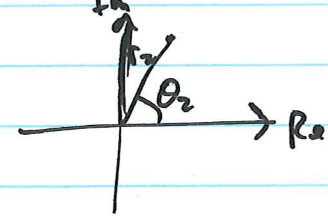
$$\theta = \operatorname{Arg}(z) = \operatorname{atan2}(b, a)$$

$$z = a + ib = |z|(\cos\theta + i\sin\theta) = \boxed{|z| e^{i\theta}}$$

Polar form is useful for multiplication and division (and other things).



$$z = r_1 e^{i\theta_1} \qquad u = r_2 e^{i\theta_2}$$



$$uz = r_1 e^{i\theta_1} r_2 e^{i\theta_2}$$

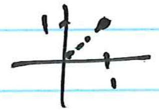
$$= \underbrace{r_1 r_2}_{\text{mult. moduli}} e^{\overbrace{i(\theta_1+\theta_2)}^{}}$$

add args

$$1/z = \frac{1}{r_1 e^{i\theta_1}} = \frac{1}{r_1} e^{-i\theta_1} \qquad \frac{u}{z} = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)}$$

Ex

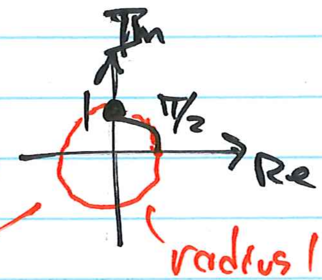
$$1+i = \sqrt{2}\, e^{i\pi/4}$$
$$1-i = \sqrt{2}\, e^{-i\pi/4}$$



$$\frac{1+i}{1-i} = \frac{\cancel{\sqrt{2}}\, e^{i\pi/4}}{\cancel{\sqrt{2}}\, e^{-i\pi/4}} = e^{i\pi/2} = i$$

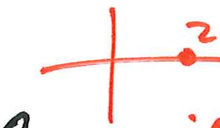
$$= 1 e^{i\pi/2}$$

$$= 0 + 1i$$



Ex: Find all three roots of $z^3 = 2$ (the cube roots of 2)

look for the answer z in polar form $z = re^{i\theta}$



We want $2 = z^3 = r^3(e^{i\theta})^3 = r^3 e^{i3\theta} = 2e^{i0}$

$$\Rightarrow r^3 = 2, \quad 3\theta = 0$$

$$\boxed{r_1 = \sqrt[3]{2} \approx 1.26} \leftarrow \boxed{r = \sqrt[3]{2}, \ \theta = 0}$$

real number

We have 1 root. Can also write

$$2 = 2e^{i(0+2\pi)} = 2e^{i2\pi}$$

or any multiple of 2π

Now: $r^3 e^{i3\theta} = 2e^{i2\pi}$ $\quad r = \sqrt[3]{2}, \ 3\theta = 2\pi$

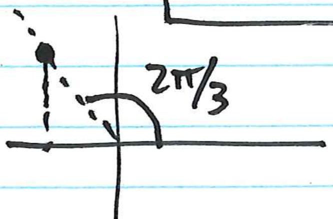
$$\theta = \frac{2\pi}{3}$$

$$\boxed{r_2 = \sqrt[3]{2}\, e^{i2\pi/3} = \cancel{-\frac{\sqrt{3}}{2} + \frac{i}{2}} \ \cancel{\approx -0.866 + 0.5i}}$$

wrong in lecture

$$= \sqrt[3]{2}\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = \sqrt[3]{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

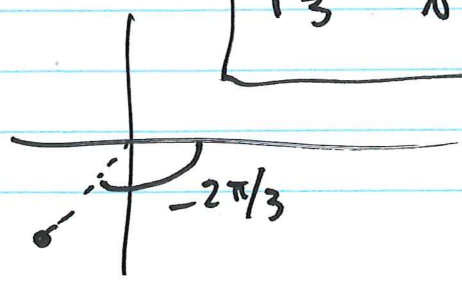
$$\approx -0.630 + 1.091i$$



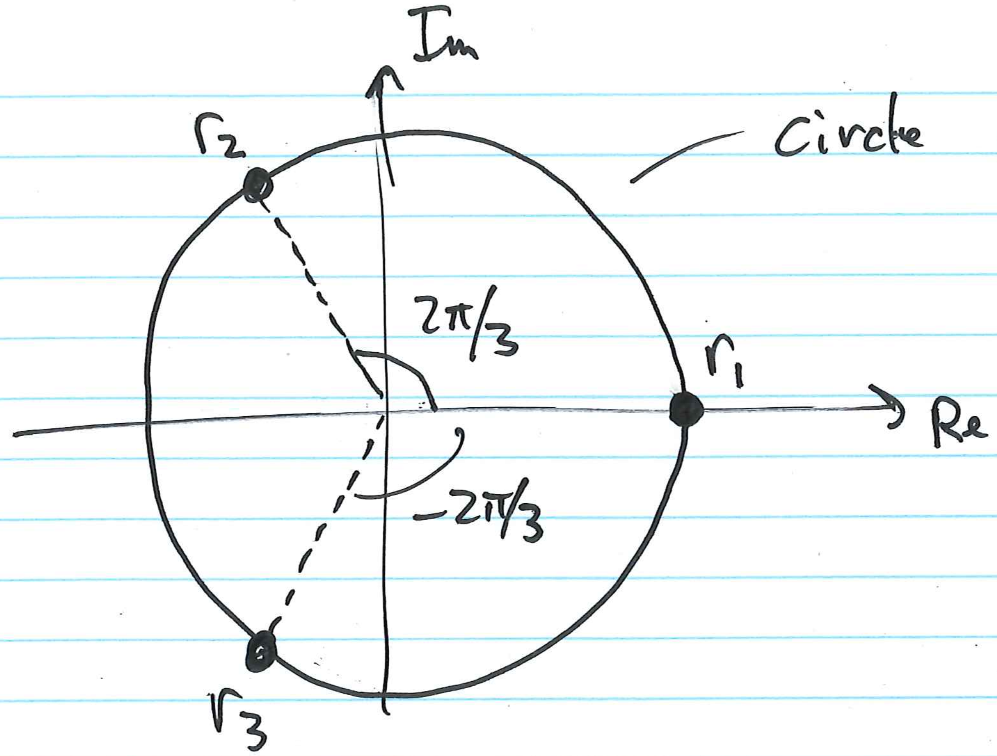
Finally $r^3 e^{i3\theta} = 2e^{-i2\pi}$ $\quad r = \sqrt[3]{2}, \ 3\theta = -2\pi$

$$\theta = -\frac{2\pi}{3}$$

$$\boxed{r_3 = \sqrt[3]{2}\, e^{-i2\pi/3} \ \cancel{= -\frac{\sqrt{3}}{2} - \frac{i}{2} \approx -0.866 - 0.5i}}$$



$$\approx -0.630 - 1.091i$$



Circle of radius $\sqrt[3]{2}$

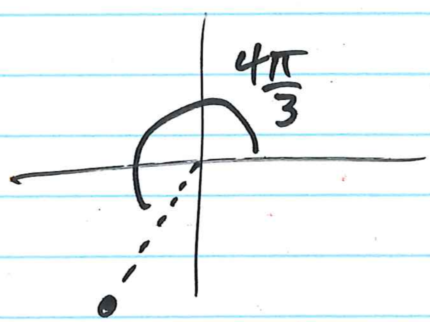
≈ 1.26

__Note__ : what if I put 4π?

$$r^3 e^{i3\theta} = 2e^{i4\pi}$$

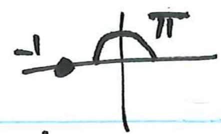
$$3\theta = 4\pi$$

$$\theta = \frac{4\pi}{3}$$



But this is just r_3 again

__Note__ : $\theta = \text{Arg}(z)$ gives $-\pi < \theta \leq \pi$



Ex Find the roots of $z^4 = -1$

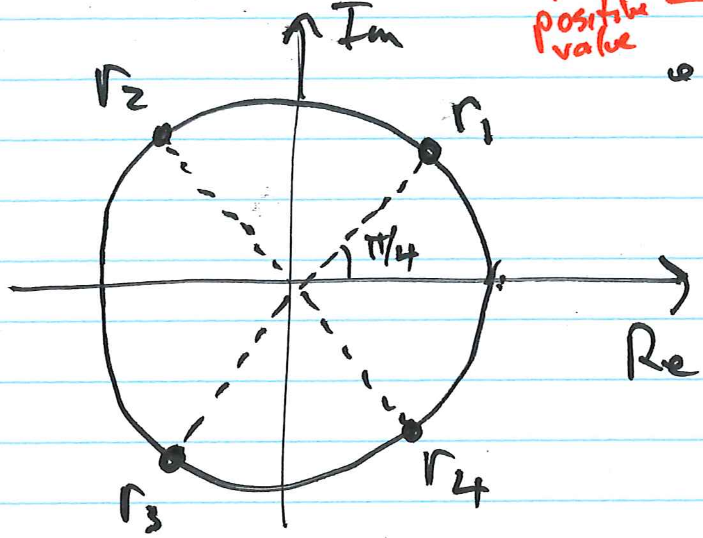
$$z = \underset{\mathbb{R}}{r}\, e^{i\underset{\mathbb{R}}{\theta}} \qquad r^4 e^{i4\theta} = -1 = 1\cdot e^{i\pi}$$

take positive value → $\boxed{\begin{array}{l} r^4 = 1 \\ r = 1 \end{array}}$ $\qquad 4\theta = \pi$, $\quad \theta = \pi/4$

...

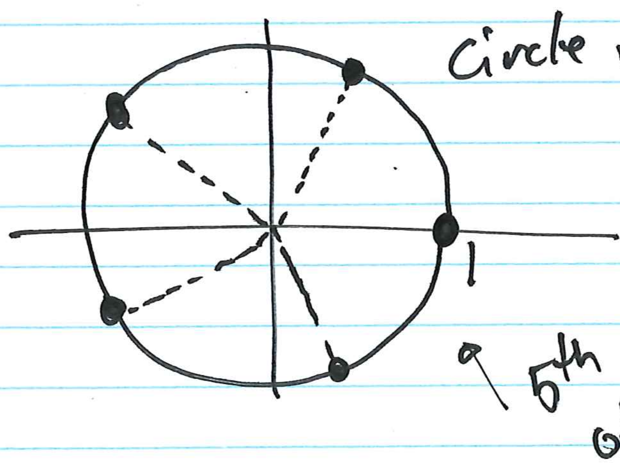
$$4\theta = \pi,\ 3\pi,\ 5\pi,\ 7\pi$$

$$\theta = \underset{r_1}{\frac{\pi}{4}},\ \underset{r_2}{\frac{3\pi}{4}},\ \underset{r_3}{\frac{5\pi}{4}},\ \underset{r_4}{\frac{7\pi}{4}}$$



Note polynomial z^4+1 has no real roots but have 4 complex roots as above.

Ex n roots of unity : solutions of $z^n = 1$ (unit)



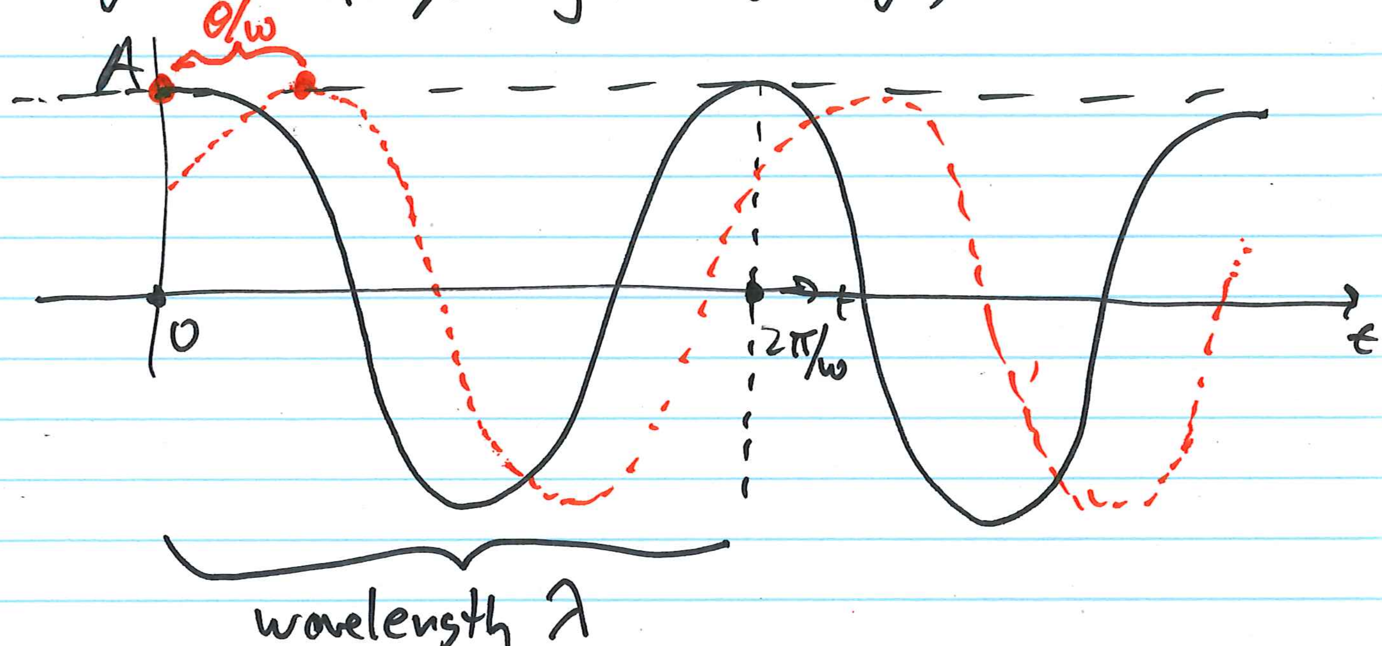
Circle radius 1

$r_j = e^{i2\pi j/n}$, $j = 0, \dots, n-1$

5th roots of unity

3 representations of a sinusoidal signal

$A\cos(\omega t)$ is signal (fcn of time) with ~~ampt~~ amplitude A and angular frequency ω (omega)



"regular" frequency is $f = \omega / 2\pi$

wavelength $\lambda = 1/f = 2\pi/\omega$

phase shift

More general: $A\cos(\omega t - \theta)$ (1)

$-\pi < \theta < \pi$

Using trig id: $c_1 \cos \omega t + c_2 \sin \omega t$ (2)

Equivalent $A = \sqrt{c_1^2 + c_2^2}$, $\theta = \text{atan2}(c_2, c_1)$ $\Longleftrightarrow$ $c_1 = A\cos\theta$, $c_2 = A\sin\theta$

We can also use exponential function:

$$A\cos(\omega t-\theta) = \text{Re}\{A e^{i(\omega t-\theta)}\}$$
$$= \text{Re}\{\underbrace{Ae^{-i\theta}}\, e^{i\omega t}\}$$

complex amplitude

$$C = Ae^{-i\theta}$$

note $|C| = A$

$$\boxed{\text{Re}\{C e^{i\omega t}\}} \quad (3)$$

equivalent to (1) and (2)

$C = Ae^{-i\theta}$ has both real amplitude A and the phase shift θ

Eg Show $u(t) = a\cos\omega t + b\sin\omega t$

solves the D.E. $\dfrac{u''}{u} = -\omega^2$

easy, just differentiate but annoying to keep track of signs of sin/cos

Much easier: $u(t) = \text{Re}(\underbrace{Ce^{i\omega t}}_{v(t)})$

show $v(t)$ solves the D.E.