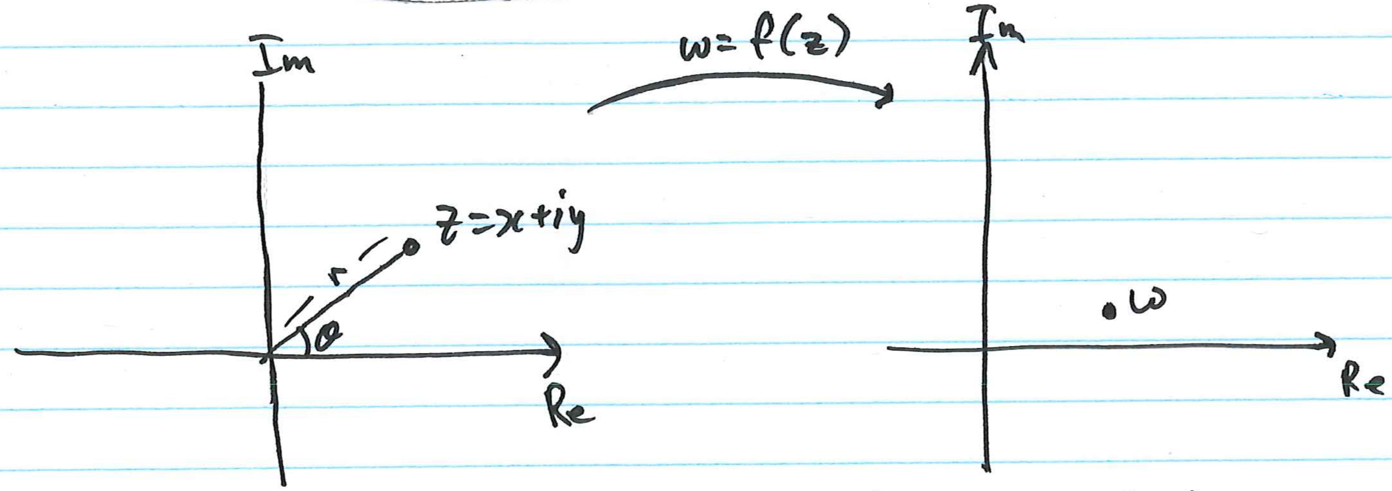
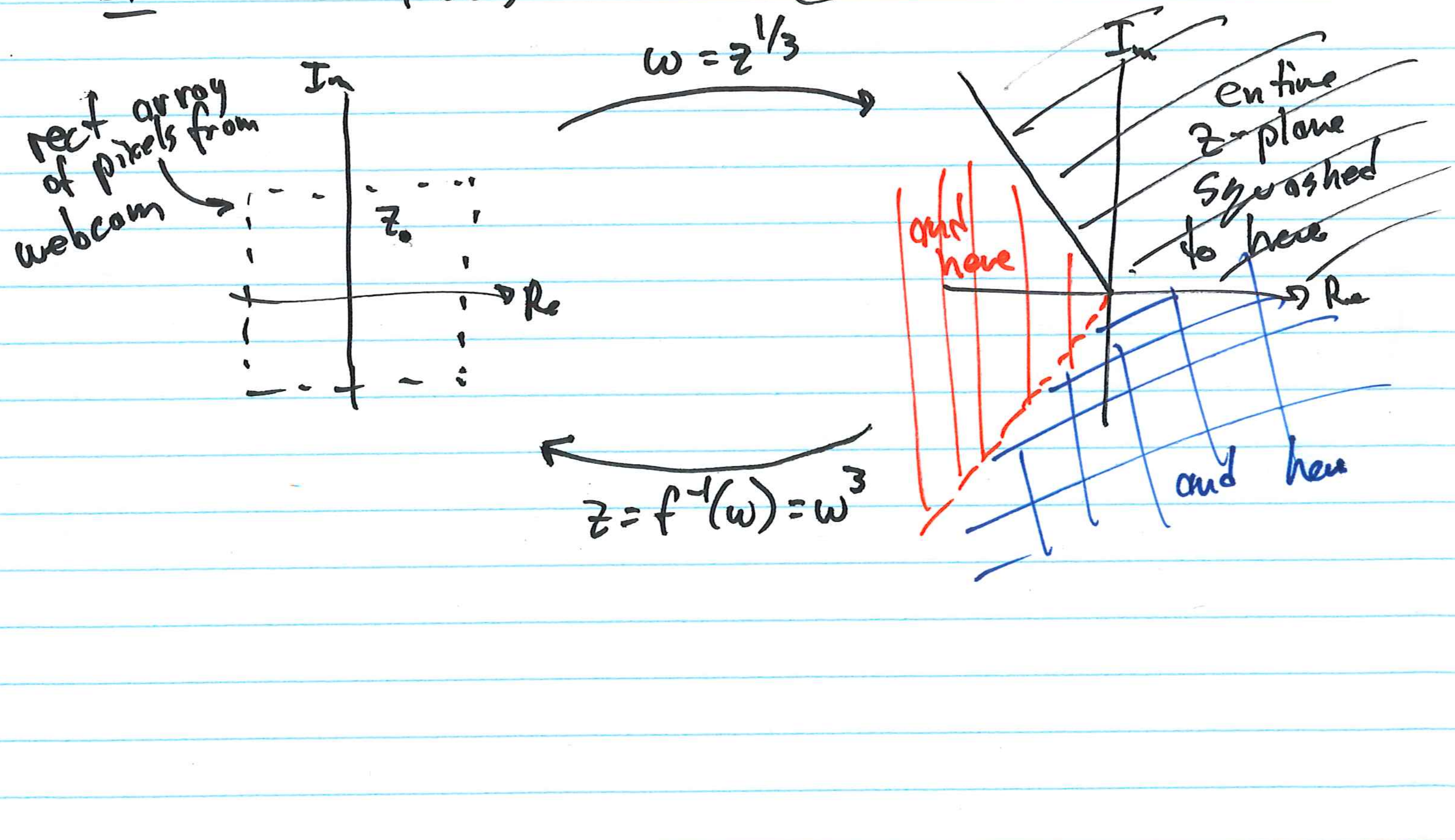
Math 152 Lecture 18

Last week: complex numbers $z = x + iy = re^{i\theta}$

<u>Functions of complex numbers</u>: $w = f(z)$



<u>Ex</u> $\quad w = f(z) = z^{1/3} = (re^{i\theta})^{1/3} = r^{1/3} e^{i\theta/3}$



§6 ~~The~~ Eigenvalues and Eigenvectors

Motivation: problem we saw e.g., in random walks was $A^n \vec{x}$ for given $m \times m$ matrix A, vector $\vec{x}$ and integer n (large?)

Ex $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Play w/ this matrix, can we understand $A^n \vec{x}$ for special values of $\vec{x}$?

$\det(A) = 0$ (b/c row 3 = row 1)

↳ A not invertible

↳ $A\vec{x} = \vec{0}$ (homog. prob.) has nontrivial sol'ns

↳ find them

$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (3) − (1)

$x_3 = t$
$x_2 = 0$
$x_1 = -t$

So $\vec{x} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ solves $A\vec{x} = \vec{0}$ for any t.

In particular $A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

multiply $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ by A is same as multiplying by the scalar 0.

Q: any other vectors like this?

$$A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}4\\2\\4\end{bmatrix}$$ so for $\vec{x} = [1, 2, 3]^T$, $A\vec{x}$ is <u>not</u> a multiple of $\vec{x}$.

<u>But</u> $A\begin{bmatrix}4\\2\\4\end{bmatrix} = \begin{bmatrix}8\\2\\8\end{bmatrix}$, $A\begin{pmatrix}8\\2\\8\end{pmatrix} = \begin{bmatrix}16\\2\\16\end{bmatrix}$ ....

<u>Try</u>: $A\begin{bmatrix}1\\0\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix}$ mult by 2 $\qquad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ mult by 1

<u>Def'n</u> An <u>eigenvector</u> $\vec{x}$ of A is a <u>nonzero</u> vector such that

$$A\vec{x} = \lambda\vec{x}$$

for some constant λ, called the <u>eigenvalue</u> (corresponding to that eigenvector).

<u>Ex</u> ~~In~~ For the matrix above, we ~~have~~ saw

$$\vec{v}_1 = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

were eigenvectors of A with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$. That is

$$A\vec{v}_1 = \vec{0}, \quad A\vec{v}_2 = \vec{v}_2, \quad A\vec{v}_3 = 2\vec{v}_3$$

Ex Compute $A^n \vec{x} = AA \dots A\vec{x}$ where $\vec{x}$ is an eigenvector.

- $A^n \vec{v}_1 = A^{n-1}(A\vec{v}_1) = A^{n-1}\vec{0} = \vec{0}$
- $A^n \vec{v}_2 = A^{n-1}(A\vec{v}_2) = A^{n-1}\vec{v}_2 = A^{n-2}(A\vec{v}_2)$
$= A^{n-2}\vec{v}_2$
$= \dots = \vec{v}_2$
- $A^n \vec{v}_3 = A^{n-1}(A\vec{v}_3) = A^{n-1}(2\vec{v}_3) = \dots = 2^n \vec{v}_3$

In general: $A\vec{x} = \lambda \vec{x}$

$$\Rightarrow A^n \vec{x} = \lambda^n \vec{x}$$

Ex Note $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. indep. (not shown). So a basis for $\mathbb{R}^3$. Recall ~~$A\vec{x}$ where~~ $\vec{x} = [1\ 2\ 3]^T$ was <u>not</u> an eigenvector. <u>but</u>

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 2\vec{v}_3$$

So
$$A^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A^n \vec{v}_1 + A^n(2\vec{v}_2) + A^n(2\vec{v}_3)$$
$$= \vec{0} + \underset{(A^n\vec{v}_2 = \vec{v}_2)}{2\vec{v}_2} + \underset{(A^n\vec{v}_3 = 2^n\vec{v}_3)}{2^{n+1}\vec{v}_3}$$
$$= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2^{n+1} \\ 0 \\ 2^{n+1} \end{bmatrix} = \begin{bmatrix} 2^{n+1} \\ 2 \\ 2^{n+1} \end{bmatrix}$$

Theory: $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$

- if $\vec{x} \neq \vec{0}$ is an eigenvector with eigenvalue λ, then $t\vec{x}$, $t \neq 0$ is also an eigenvector with the same eigenvalue λ. ONLY DIRECTION MATTERS

  Proof: $A(t\vec{x}) = tA\vec{x}$
  $= t(\lambda\vec{x}) = (t\lambda)\vec{x} = (\lambda t)\vec{x}$
  $= \lambda(t\vec{x})$

- $\vec{x} \neq \vec{0}$ (by def'n) but $\lambda = 0$ is possible.

  if $\lambda = 0$ is an eigenvalue then
  $A\vec{x} = \cancel{\lambda\vec{x}} = \vec{0}$
  has nontrivial sol'ns. $\Rightarrow$ A not invertible
  $\Rightarrow \det(A) = 0$, etc.

A invertible : A does <u>not</u> have a $\lambda = 0$ eigenvalue.

- $A\vec{x} = \lambda\vec{x}$: <u>matrix</u> <u>mult</u> of an eigenvector becomes <u>scalar</u> <u>mult</u> by the <u>eigenvalue</u>.

- $A^n\vec{x} = \lambda^n\vec{x}$.

How to find eigenvalues λ of a $m \times m$ matrix A

$$A\vec{x} = \lambda\vec{x} \qquad \text{for } \vec{x} \neq \vec{0}$$

$$A\vec{x} = \lambda I \vec{x}$$

$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\boxed{(A - \lambda I)\vec{x} = \vec{0}}$$

we want this to have nonzero sol'ns $\vec{x}$

so we need:

$$\det(A - \lambda I) = 0 \qquad (*)$$

find λ which make this $\det = 0$

Turns out $(*)$ is a (symbolic) polynomial in λ of degree ~~at most~~ m, with m roots (repeated, complex). Each root is an eigenvalue.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ \cancel{0} & \cancel{1-\lambda} & \cancel{0} \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -0\det(\quad) + (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} - 0\det(\,)$$

$$= (1-\lambda)\left((1-\lambda)^2 - 1\right) = \text{cubic polynomial}$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) = (1-\lambda)\lambda(\lambda - 2)$$

Solving $\det(A - \lambda I) = 0$

$$\lambda(1-\lambda)(\lambda-2) = 0$$

roots: $\lambda = 0$, $\lambda = 1$, $\lambda = 2$ (only roots)

$$\boxed{\text{The eigenvalues are } \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2}$$

<u>Ex</u> find the corresponding <u>eigenvectors</u>

↳ look for homog. solns of $(A - \lambda I)\vec{x} = \vec{0}$

$\boxed{\lambda_1 = 0}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

soln $\vec{v}_1 = t\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$

only direction matters so take $t = 1$

$$\boxed{\vec{v}_1 = [-1, 0, 1]^T \text{ with } \lambda_1 = 0}$$

$\boxed{\lambda_2 = 1}$

$$A - \lambda_2 I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = s\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

$$\boxed{\vec{v}_2 = [0, 1, 0]^T \text{ with } \lambda_2 = 1}$$

$\boxed{\lambda_3 = 2}$

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \underset{(3)+(1)}{\sim} \begin{bmatrix} \boxed{-1} & 0 & 1 \\ 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = t \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$$

$$\boxed{\text{so } \vec{v}_3 = [1, 0, 1]^T \text{ with } \lambda_3 = 2}$$

Recipe: (1) $\det(A - \lambda I) = 0$ — scalar poly equation

for m roots.

(2) solve $(A - \lambda_i I)\vec{x}_i = \vec{0}$

to find the corresponding eigenvectors.

solving m different linear algebra problems.

GE m times