Math 152 Lecture 20

~~Last~~ Last week: eigenvalues / eigenvectors

Given a $m\times m$ matrix A, look for $\vec{x}$, λ that solve $\boxed{A\vec{x} = \lambda\vec{x}}$

$\vec{x} \neq \vec{0}$

"matrix · vector = scalar · vector"

<u>Ex</u> $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

a) find evals/evecs

b) write $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as lin. comb. of the evecs

c) evaluate $A^{10}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

a) Recall: $A\vec{x} - \lambda\vec{x} = \cancel{A}\ \cancel{AI}\ \underbrace{(A-\lambda I)}_{\det(A-\lambda I)=0}\vec{x} = \vec{0}$? $? \neq \vec{0}$

solve for λ

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda = 1, 3}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \vec{x} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$\underline{\lambda_1 = 1}$ $\qquad A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix}$

solve: $(A - \lambda_1 I)\vec{x} = \vec{0}$

(homog problem)

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 3 \qquad A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$ Columns consisting of eigenvectors

$$\left[\begin{array}{cc|c} -1 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \end{array}\right]$$

$$\vec{c} = [1/2, \ 5/2]^T$$

so $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2$

c) $A^{10}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2}\underbrace{A^{10}\vec{v}_1}_{\lambda_1^{10}\vec{v}_1} + \frac{5}{2}\underbrace{A^{10}\vec{v}_2}_{\lambda_2^{10}\vec{v}_2} = \frac{1}{2}(1)^{10}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2}(3)^{10}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{5}{2}3^{10} - 1/2 \\ \frac{5}{2}3^{10} + 1/2 \end{bmatrix}$$

largest eigenvalue dominates

3

solve $T\vec{c} = \vec{x}$

where $T = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$

Recipe for $A^n \vec{x}$

(a) find evals / evecs

$\lambda_1, \ldots, \lambda_m$ $\qquad$ $C = \{\vec{v}_1, \ldots, \vec{v}_m\}$

(b) Write $\vec{x}$ as lin. comb. of C (in Math 152, C will always be a basis for $\mathbb{R}^m$).

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_m \vec{v}_m$$

(c) $A^n \vec{x} = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2 + \ldots + c_m \lambda_m^n \vec{v}_m$

Ex Find formula for the entries of the Fibonnocci sequence:

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \ldots$$

$$f_n = f_{n-1} + f_{n-2}$$

row1: $f_{n+2} = f_{n+1} + f_n$

Modelling: $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ $\qquad$ $\vec{x}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}_n$

$$\boxed{\vec{x}_{n+1} = A\vec{x}_n} = A(A\vec{x}_{n-1}) = A^2 \vec{x}_{n-1} = A^3 \vec{x}_{n-2}$$

~~$\vec{x}_n = A^n \cancel{\vec{x}_0} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$~~ $\quad \vec{x}_1$

$$\boxed{\vec{x}_n = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \underline{\text{or}} \quad \vec{x}_n = A^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{n=2}$ $\quad \vec{x}_2 = \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = A \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = A^2 \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = A^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Recipe a) $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2-\lambda-1$

$= (\lambda-\lambda_1)(\lambda-\lambda_2)$

quadratic formula: $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

note here: $\frac{1}{\lambda_1} = -\lambda_2$

$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $A-\lambda_1 I = \begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix} \sim \begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}$

[0 0] must be here, without doing GE.

$\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$ (check)

Similarly, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, $\vec{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

b) solve $\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow c_1 = 1/\sqrt{5}, \quad c_2 = -\frac{1}{\sqrt{5}}$

c) $\vec{x}_n = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2$

$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

f_n = 2nd entry of $\vec{x}_n$

$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ = integer!

Repeated eigenvalues

Ex $A = I_2$ evals/evecs: $I\vec{x} = \vec{x} = 1\vec{x}$

evals $\downarrow$ $\lambda = 1$

evecs $\downarrow$ any nonzero vector

Compute: (1) $\det(I - \lambda I) \overset{\text{def}}{=} \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2$

$\lambda_1 = 1, \ \lambda_2 = 1$

(2) $\lambda = 1$ $(I - 1I)\vec{x} = \vec{0}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$\vec{x} = s\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (or indeed any two lin indep vectors)

$\lambda = 1$ has a two-dimensional space of eigenvectors.

$\lambda_1 = 1, \ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 1, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Ex $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)^2$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 1$ (repeated)

evecs

$\lambda = 1$, $A - I = \begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$ pivot $\vec{x} = \begin{bmatrix} s \\ 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

only 1 eigenvector (!!)

In this example, we do not have a basis of eigenvectors (need two l.i. vectors, but have only one!!

↳ A is a "defective matrix"

↳ no more of these in Math 152

↳ fix is "Jordan Form" (not covered)

$$\left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$

$1x_2 = 0$

$x_1 = t$

Ex $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda(1-\lambda)$$

$\lambda_1 = \lambda_2 = 1$, $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\vec{x}_2 = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_3 = 2$

$A - 2I \Rightarrow \ldots \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$