Math 152 Lecture 21 Complex Eigenvalues

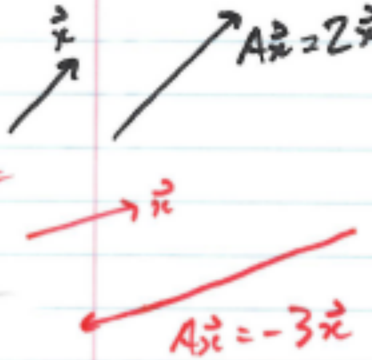
Ex ~~R~~ in 2D, let $R = \mathrm{Rot}_\theta$, $\theta \neq 0, \pi$

$R\vec{x}$ rotates $\vec{x}$ by $\theta \Rightarrow R\vec{x} \neq \lambda \vec{x}$, $\vec{x} \neq 0$, $\lambda \in \mathbb{R}$

↳ no real eigenvals

Aside

$A\vec{x} = \lambda \vec{x}$



Geometric argument

Algebra: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\det(R - \lambda I) = \det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$= (\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$(\lambda - \cos\theta)^2 = -\sin^2\theta \Rightarrow \lambda - \cos\theta = \pm\sqrt{-1}\,\sin\theta$

$$\boxed{\lambda = \cos\theta \pm i \sin\theta}$$

$\lambda_1 = \cos\theta + i\sin\theta$: $A - \lambda I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \sim \begin{bmatrix} -i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}$

(div by $\sin\theta \neq 0$) $\sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = \cancel{s}\ 1$ (free)

$i x_1 + 1 = 0$

$x_1 = -1/i = \frac{-i}{i^2} = i$

$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Could repeat for $\lambda_2 = \cos\theta - i\sin\theta$ or use $\lambda_2 = \bar{\lambda}_1$

Note: for real matrix $\overline{A\vec{v}} = \overline{\lambda\vec{v}} \Rightarrow \bar{A}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}} \Rightarrow A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$
so $\bar{\lambda}$ and $\bar{\vec{v}}$ also an eval/evec.

In our case $\lambda_2 = \cos\theta - i\sin\theta$, $\vec{v}_2 = \bar{\vec{v}}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Ex $A=\begin{bmatrix}0&0&3\\1&0&-1\\0&1&3\end{bmatrix}$, find evals/evecs, hint: $\lambda_1=3$

$$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda&0&3\\1&-\lambda&-1\\0&1&3-\lambda\end{bmatrix} = -\lambda(-\lambda(3-\lambda)+1)+0+3(1)$$
$$= -\lambda^3+3\lambda^2-\lambda+3=0 \; *$$

hint $\lambda_1=3$ is a root: divide ~~by~~ my polynomial by $\lambda-3$

$$\begin{array}{r|l} & -\lambda^2 \qquad -1 \\ \hline \lambda-3 & -\lambda^3+3\lambda^2-\lambda+3 \\ & \ominus\; -\lambda^3+3\lambda^2 \\ \hline & \qquad\quad -\lambda+3 \\ & \qquad \ominus\; -\lambda+3 \\ \hline & \qquad\qquad 0 \end{array}$$

polynomial long div

$$\Rightarrow (\lambda-3)(-\lambda^2-1)=0 \; *$$

$\lambda_1=3$ $\qquad$ $\lambda^2+1=0$

$\lambda_{2,3}=\pm i$

$\lambda_1=3$: $A-\lambda I$: $\begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}$ $\qquad \vec{v}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$

$\lambda_2=i$: $A-iI=\begin{bmatrix}-i&0&3\\1&-i&-1\\0&1&3-i\end{bmatrix}$ $\begin{matrix} i(1) \\ (2)\leftrightarrow(3)\end{matrix}$ $\sim \begin{bmatrix}1&0&3i\\0&1&3-i\\1&-i&-1\end{bmatrix}$ ← will eventually go to 0 row

$\vec{v}_2=\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}$ $\qquad$ $x_1=-3i(1)$, $x_2=-(3-i)(1)$, $x_3=\text{free}=1$ $\qquad \sim \begin{bmatrix}\boxed{1}&0&3i\\0&\boxed{1}&3-i\\0&0&0\end{bmatrix}$

$\lambda_3=-i=\overline{\lambda_2}$ $\qquad$ so $\vec{v}_3=\overline{\vec{v}_2}=\begin{bmatrix}\overline{-3i}\\\overline{-3+i}\\\overline{1}\end{bmatrix}=\begin{bmatrix}3i\\-3-i\\1\end{bmatrix}$

③

should be $\frac{1}{10}$

Ex with previous A, evaluate $A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix}$

step 1 : find evals/evecs

step 2 : $T\vec{c} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ solve for lin. comb. of evecs

$$\begin{bmatrix}1 & -3i & 3i\\0 & -3+i & -3-i\\1 & 1 & 1\end{bmatrix}\vec{c} = \begin{bmatrix}1\\0\\0\end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c}1 & -3i & 3i & 1\\0 & -3+i & -3-i & 0\\1 & 1 & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\0 & 1 & 0 & -\frac{1}{20}+\frac{3}{20}i\\0 & 0 & 1 & -\frac{1}{20}-\frac{3}{20}i\end{array}\right]$$

c_2, c_3 complex conj

so $\begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{v}_1 + \left(-\frac{1}{20}+\frac{3}{20}i\right)\vec{v}_2 + \left(-\frac{1}{20}-\frac{3}{20}i\right)\vec{v}_3$

$$A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix} = c_1\lambda_1^{13}\vec{v}_1 + c_2\lambda_2^{13}\vec{v}_2 + c_3\lambda_3^{13}\vec{v}_3$$

$$= {}_{\wedge}^{0.1}\, 3^{13}\begin{bmatrix}1\\0\\1\end{bmatrix} + \underbrace{\left(-\frac{1}{20}+\frac{3}{20}i\right)i^{13}\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}}_{\vec{U}+\vec{V}} + \underbrace{\left(-\frac{1}{20}-\frac{3}{20}i\right)(-i)^{13}\begin{bmatrix}3i\\-3-i\\1\end{bmatrix}}_{\vec{U}-\vec{V}}$$

note: $i^{13} = i^{12}i = (-1)^6 i = i$

$2\,\mathrm{Re}\{2^{nd} \text{ term}\}$

$$\mathrm{Re}\left\{i\left(-\frac{1}{20}+\frac{3}{20}i\right)\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}\right\}$$

$$= \mathrm{Re}\left\{\left(-\frac{3}{20}-\frac{1}{20}i\right)\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}\right\} = \begin{bmatrix}-3/20\\1/2\\-3/20\end{bmatrix}$$

missing $\frac{1}{10}$ last day

$$A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0.1\cdot 3^{13} - 6/20\\1\\0.1\cdot 3^{13} - 6/20\end{bmatrix}$$

todo: check on computer!

Note: λ^{13} was easy here b/c $\lambda = i$, more generally you may want use polar form.
e.g. $\lambda = 1+i = \sqrt{2}\, e^{i\pi/4}$

$$\lambda^{13} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{13} = 2^{13/2} e^{i\,13\pi/4}$$
$$= 2^{13/2} e^{i\,5\pi/4}$$
$$= 2^{13/2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$
$$= 2^{6}(-1-i) = -64-64i$$

Theory ▪ complex evals/evecs of real matrices come in complex conj pairs

$(\lambda, \vec{v})$ ↳ $(\bar{\lambda}, \bar{\vec{v}})$

▪ the dimension of space eigenvectors corresponding to λ is at least one and at most the multiplicity of the root λ ~~order~~ of $\det(A-\lambda I)$

examples last day.

▪ if $C = \{v_1, \ldots, v_m\}$ is a set of evecs corresponding to different evals then C is l.i. → "a basis of eigenvectors"

↳ if A has no repeated evals, it has a basis of eigenvectors

↳ if A has repeated eval, there may or may not be a basis of evecs.

↳ possible to modify recipe for $A^n \vec{x}$ but not in this course.

common in many applications

* if A is symmetric then it always has real eigenvalues and it always has a basis of eigenvecs (even if there are repeated roots).

Random walks recall: problem $P^n \vec{x}^{(0)}$

P: prob. transition matrix — non-negative P_{ij}, columns sum to 1

$\vec{x}^{(0)}$: initial prob. vec.

defn If $\lim_{n\to\infty} P^n \vec{x}^{(0)} = \vec{p}$ for every initial prob vec $\vec{x}^{(0)}$, then we say the random walk has an equilibrium probability vector $\vec{p}$

Theory for transition matrix P:

(1) eigenvalues satisfy $|\lambda| \le 1$

(2) always at least one eigenvalue $\lambda_1 = 1$. If this is the only eigenvalue with $|\lambda| = 1$ then $\vec{v}_1$ (evec with $\lambda_1 = 1$) is (after scaling) the equilibrium probability

(3) if $|\lambda| < 1$, then the corresponding evec has entries that sum to zero.

(4) if P (or power of P) is matrix with strictly positive entries, the random walk has an equilibrium probability

Ex $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ by (4) it has an equilibrium prob.