Math 152 Lecture 21 Complex Eigenvalues

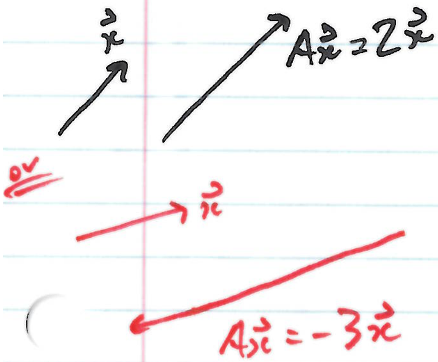
Ex ~~Rz~~ in 2D, let $R = \text{Rot}_\theta$, $\theta \neq 0, \pi$

$R\vec{x}$ rotates $\vec{x}$ by $\theta \Rightarrow R\vec{x} \neq \lambda\vec{x}$, $\vec{x} \neq 0$, $\lambda \in \mathbb{R}$

↳ no real eigenvals

Aside

$A\vec{x} = \lambda\vec{x}$



Geometric argument

Algebra: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det(R - \lambda I) = \det\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$

$= (\lambda - \cos\theta)^2 + \sin^2\theta = 0$

$(\lambda - \cos\theta)^2 = -\sin^2\theta \Rightarrow \lambda - \cos\theta = \pm\sqrt{-1}\,\sin\theta$

$$\boxed{\lambda = \cos\theta \pm i\sin\theta}$$

$\lambda_1 = \cos\theta + i\sin\theta$: $\quad A - \lambda I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \sim \begin{bmatrix} -i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}$

(div by $\sin\theta \neq 0$) $\sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = \cancel{s}\ 1$ (free

$i x_1 + 1 = 0$

$x_1 = -1/i = \frac{-i}{i^2} = i$

$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Could repeat for $\lambda_2 = \cos\theta - i\sin\theta$ or use $\lambda_2 = \bar{\lambda}_1$

* Note: for real matrix $\overline{A\vec{v}} = \overline{\lambda\vec{v}} \Rightarrow \bar{A}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}} \Rightarrow A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

so $\bar{\lambda}$ and $\bar{\vec{v}}$ also an eval/evec.

In our case $\lambda_2 = \cos\theta - i\sin\theta$, $\vec{v}_2 = \bar{\vec{v}}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Ex $A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, find evals/evecs, hint: $\lambda_1 = 3$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 3 \\ 1 & -\lambda & -1 \\ 0 & 1 & 3-\lambda \end{bmatrix} = -\lambda(-\lambda(3-\lambda)+1) + 0 + 3(1)$$

$$= -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0^{*}$$

hint $\lambda_1 = 3$ is a root: divide ~~by~~ my polynomial by $\lambda - 3$

$$\begin{array}{r} -\lambda^2 \quad\; -1 \\ \lambda - 3 \,\overline{)\, -\lambda^3 + 3\lambda^2 - \lambda + 3} \\ \ominus \; \underline{-\lambda^3 + 3\lambda^2 \qquad\quad} \\ -\lambda + 3 \\ \ominus \; \underline{-\lambda + 3} \\ 0 \end{array}$$

polynomial long div

$$\Longrightarrow \quad (\lambda - 3)(-\lambda^2 - 1) = 0^{*}$$

$\lambda_1 = 3$ $\qquad$ $\lambda^2 + 1 = 0$

$\lambda_{2,3} = \pm i$

$\lambda_1 = 3$: $A - \lambda I$: $\begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = i$: $A - iI = \begin{bmatrix} -i & 0 & 3 \\ 1 & -i & -1 \\ 0 & 1 & 3-i \end{bmatrix}$ $\begin{array}{l} i(1) \\ (2) \leftrightarrow (3) \end{array}$ $\sim \begin{bmatrix} 1 & 0 & 3i \\ 0 & 1 & 3-i \\ 1 & -i & -1 \end{bmatrix}$ ← will eventually go to 0 row

$\vec{v}_2 = \begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix}$ $\qquad \begin{array}{l} x_1 = -3i(1) \\ x_2 = -(3-i)(1) \\ x_3 = \text{free} = 1 \end{array}$ $\qquad \sim \begin{bmatrix} \boxed{1} & 0 & 3i \\ 0 & \boxed{1} & 3-i \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda_3 = \cancel{\lambda_2}\, -i = \overline{\lambda_2}$ $\qquad$ so $\vec{v}_3 = \overline{\vec{v}_2} = \begin{bmatrix} \overline{-3i} \\ \overline{-3+i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} 3i \\ -3-i \\ 1 \end{bmatrix}$

Ex with previous A, evaluate $A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix}$

step 1 : find evals/evecs

step 2 : $T\vec{c} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ solve for lin. comb. of evecs

$$\begin{bmatrix}1 & -3i & 3i\\ 0 & -3+i & -3-i\\ 1 & 1 & 1\end{bmatrix}\vec{c} = \begin{bmatrix}1\\0\\0\end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c}1 & -3i & 3i & 1\\ 0 & -3+i & -3-i & 0\\ 1 & 1 & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -\frac{1}{20}+\frac{3}{20}i\\ 0 & 0 & 1 & -\frac{1}{20}-\frac{3}{20}i\end{array}\right]$$

c_2, c_3 complex conj

so $\begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{v}_1 + \left(-\frac{1}{20}+\frac{3}{20}i\right)\vec{v}_2 + \left(-\frac{1}{20}-\frac{3}{20}i\right)\vec{v}_3$

$$A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix} = c_1\lambda_1^{13}\vec{v}_1 + c_2\lambda_2^{13}\vec{v}_2 + c_3\lambda_3^{13}\vec{v}_3$$

$$= 3^{13}\begin{bmatrix}1\\0\\1\end{bmatrix} + \underbrace{\underbrace{\left(-\frac{1}{20}+\frac{3}{20}i\right)i^{13}\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}}_{\vec{U}+\vec{V}} + \underbrace{\left(-\frac{1}{20}-\frac{3}{20}i\right)(-i)^{13}\begin{bmatrix}3i\\-3-i\\1\end{bmatrix}}_{\vec{U}-\vec{V}}}_{2\,\mathrm{Re}\{2^{nd}\text{ term}\}}$$

note:
$i^{13} = i^{12}i = (-1)^6 i = i$

$$\mathrm{Re}\left\{i\left(-\frac{1}{20}+\frac{3}{20}i\right)\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}\right\}$$

$$= \mathrm{Re}\left\{\left(-\frac{3}{20}-\frac{1}{20}i\right)\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}\right\} = \begin{bmatrix}-3/20\\1/2\\-3/20\end{bmatrix}$$

$$A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3^{13}-6/20\\1\\3^{13}-6/20\end{bmatrix}$$

todo:
check on computer!

Note: λ^{13} was easy here b/c $\lambda = i$, more generally you may want use polar form.

e.g. $\lambda = 1 + i = \sqrt{2}\, e^{i\pi/4}$

$$\lambda^{13} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{13} = 2^{13/2} e^{i\,13\pi/4}$$
$$= 2^{13/2} e^{i\,5\pi/4}$$
$$= 2^{13/2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$
$$= 2^{6}(-1-i) = -64-64i$$

Theory ▪ complex evols/evecs of real matrices come in complex conj pairs

$(\lambda, \vec{v})$ ↳ $(\bar{\lambda}, \bar{\vec{v}})$

▪ the dimension of space eigenvectors corresponding to λ is at least one and at most the multiplicity of the root λ ~~order~~ of $\det(A - \lambda I)$

examples last day.

▪ if $C = \{v_1, \ldots, v_m\}$ is a set of evecs corresponding to different evols then C is l.i. → "a basis of eigenvectors"

↳ if A has no repeated evals, it has a basis of eigenvectors

↳ if A has repeated eval, there may or may not be a basis of evecs.

↳ possible to modify recipe for $A^n \vec{x}$ but not in this course.

common in many applications

☒ if A is symmetric then it always has real eigenvalues and it always has a basis of eigenvecs (even if there are repeated roots).

Random walks

recall: problem $P^n \vec{x}^{(0)}$

P: prob. transition matrix
- non-negative P_{ij}
- columns sum to 1

$\vec{x}^{(0)}$: initial prob. vec.

def'n
If $\lim_{n\to\infty} P^n \vec{x}^{(0)} = \vec{p}$ for every initial prob vec $\vec{x}^{(0)}$,
then we say the random walk has an equilibrium probability vector $\vec{p}$

Theory for transitition matrix P:

(1) eigenvalues satisfy $|\lambda| \le 1$

(2) always at least one eigenvalue $\lambda_1 = 1$
If this is the only eigenvalue with $|\lambda| = 1$ then $\vec{v}_1$ (evec with $\lambda_1 = 1$) is (after scaling) the equilibrium probability

(3) if $|\lambda| < 1$, then the corresponding evec has entries that sum to zero.

(4) if P (or power of P) is matrix with strictly positive entries, the random walk has an equilibrium probability

Ex $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ by (4) it has an equilibrium prob.