# Math 152 Lecture 22

~~Erratta~~ Errata last day: $1/10$ missing for $c_1$
see also lec21_ ... .m

Last day: apply eigenanalysis to random walk.

**Ex** $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ by (4) has an equilibrium prob $\vec{p}$

Eigenanalysis: evalue: $\det(A - \lambda I) = 0$ scalar eqn

$$= \det \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1/2-\lambda)(1/2-\lambda) - 1/4 = 0$$

$$\Rightarrow \lambda^2 - \lambda = 0$$

$\lambda_1 = 1$ now solve ~~$A + I x = 0$~~ $(A-I)\vec{x} = \vec{0}$ (vector eqn)

$$P - I = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{evec is } \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1$$

↳ scale to get the equilib. prob.

$$\vec{p} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

by property (2)

$\lambda_2 = 0$ $P - 0I = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

note: property (3)

Intuition behind this: let $\vec{x}^{(0)} = [a, 1-a]^T$ for some $0 \le a \le 1$

Solve $T\vec{c} = \vec{x}^{(0)}$ for $\vec{c}$ $\Rightarrow \left[\begin{array}{cc|c} 1/2 & -1 & a \\ 1/2 & 1 & 1-a \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1/2 - a \end{array}\right]$

evecs as cols

$$\vec{x}^{(0)} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = 1\vec{p} + (1/2 - a)\vec{v}_2$$

So $P^n \vec{x}^{(0)} = \underbrace{\cancel{c_1}}_{1} \underbrace{\cancel{\lambda_1^n}}_{1} \underbrace{\cancel{\vec{v}_1}}_{\vec{p}} + c_2 \underbrace{\cancel{\lambda_2^n}}_{0} \vec{v}_2 = \vec{p}$ for $n \ge 1$

**Ex** $P = \begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{bmatrix}$ Again by (4) we must have equilib prob.

Eigenanalysis: ... $\lambda_1 = 1$, $\vec{v}_1 = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$

$\lambda_2 = -\frac{1}{6}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Rescale $\vec{v}_1$ to get $\vec{p} = \frac{1}{4/3+1}\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = \frac{1}{7/3}\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} = \vec{p}}$

**Ex** same $P$, consider the walk from $\vec{x}^{(0)}$

$$\vec{x}^{(0)} = c_1 \vec{p} + c_2 \vec{v}_2$$

sum entries gives 1

entries sum to 0 by prop (3)

**Trick**: sum entries in the eqn: $1 = c_1 1 + c_2 0$

$\Rightarrow \boxed{c_1 = 1}$

So $\vec{x}^{(0)} = \underbrace{\vec{p}}_{\text{equilib prob}} + c_2 \vec{v}_2$

depends on exactly which $\vec{x}^{(0)}$

equilib prob + "perturbation" away from equilibrium

$$P^n \vec{x}^{(0)} = \vec{p} + c_2 \left(-\frac{1}{6}\right)^n \vec{v}_2$$

oscillating and <u>shrinking</u> as $n$ increases

So $\lim_{n\to\infty} P^n \vec{x}^{(0)} = \vec{p}$

**Ex** $P = \begin{bmatrix} 0 & 1/4 & 1/2 & 0 \\ 1/2 & 1/4 & 0 & 1/2 \\ 1/2 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/2 \end{bmatrix}$

note: some entries $= 0$ cannot use prop (4).

↳ Octave: $\lambda_1 = 1 \quad \vec{v}_1 = [1 \ \ 2 \ \ 1 \ \ 2]^T$

$\lambda_2 = -1/2 \quad \vec{v}_2 = [2 \ \ -2 \ \ -1 \ \ 1]^T$

$\lambda_3 = 0 \quad \vec{v}_3 = [1 \ \ -2 \ \ 1 \ \ 0]^T$

$\lambda_4 = 1/4 \quad \vec{v}_4 = [-1 \ \ 1 \ \ -1 \ \ 1]^T$

$$\vec{p} = \frac{1}{\sum_i v_{1_i}} \vec{v}_1 = \frac{1}{6}\vec{v}_1 = [1/6 \ \ 1/3 \ \ 1/6 \ \ 1/3]^T$$

$c_1 = 1$ by trick

depends on $\vec{x}^{(0)}$

Walk: $\vec{x}^{(0)} = \vec{p} + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$

$\lambda_3 = 0$

$$P^n \vec{x}^{(0)} = \vec{p} + \underbrace{c_2(-1/2)^n \vec{v}_2}_{\text{more interesting than } c_4 \text{ term b/c shrinks slower}} + c_4 (1/4)^n \vec{v}_4 + 0$$

**Ex** $P = \begin{bmatrix} 1/2 & 0 & 1/3 \\ 1/4 & 3/4 & 1/3 \\ 1/4 & 1/4 & 1/3 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1/2-\lambda & 0 & 1/3 \\ 1/4 & 3/4-\lambda & 1/3 \\ 1/4 & 1/4 & 1/3-\lambda \end{bmatrix}\begin{matrix} \times 6 \\ \times 12 \\ \times 12 \end{matrix}\right)$$

$$= \frac{1}{6\cdot 12\cdot 12} \underbrace{\det\begin{bmatrix} 3-6\lambda & 0 & 2 \\ 3 & 9-12\lambda & 4 \\ 3 & 3 & 4-12\lambda \end{bmatrix}}_{f(\lambda)} = 0$$

$f(\lambda) = -864\lambda^3 + 1368\lambda^2 - 540\lambda + 36$

But $\lambda_1 = 1$ should be a root. $f(1) = 0$ ✓

poly long div

$$
\begin{array}{r}
-864\lambda^2 + 504\lambda - 36 \\
\lambda - 1 \,\overline{\big)\, -864\lambda^3 + 1368\lambda^2 - 540\lambda + 36} \\
\ominus\; \underline{-864\lambda^2 + 864\lambda^2} \qquad\qquad \\
504\lambda^2 - 540\lambda + 36 \\
\ominus\; \underline{504\lambda^2 - 504\lambda} \qquad\quad \\
-36\lambda + 36 \\
\ominus\; \underline{-36\lambda + 36} \\
0 \quad \checkmark
\end{array}
$$

So roots are ... use the quadratic formula:

$$\lambda_1 = 1, \quad \lambda_2 = 1/2, \quad \lambda_3 = 1/12$$

Ex find the equilib prob in previous:

$$\vec{v}_1 = \begin{bmatrix} 2/3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \sum_i v_{1_i} = \frac{11}{3} \Rightarrow \vec{p} = \begin{bmatrix} 2/11 \\ 6/11 \\ 3/11 \end{bmatrix}$$

## Application of eigenanalysis: differential eqns.

**Review:** $x(t)$ is an unknown function that satisfies a differential eqn: (DE)

$$\frac{dx}{dt} = \underset{\uparrow \text{ given}}{f}(x, t) \qquad \text{for all } t. \quad (1)$$

Common in physics, biology, economics, etc

(1) is first-order, nonlinear (b/c we know nothing about $f$).

If an "initial condition" (IC) $x(0) = x_0$ is ~~given~~, given then $x(t)$ is uniquely determined (at least for many fcns $f$), at least "near" $t=0$.

DE & IC = "initial value problem" (IVP)

Simplest IVP: $\dfrac{dx}{dt} = \underset{\uparrow}{a}x, \quad x(0) = x_0 \qquad (2)$

given constant, not $a(t)$

first-order, linear, scalar, homogeneous, and constant coeff.

relax this in 152

see math 101
math 255

Review: sol'n to this IVP (2) is

$$x(t) = x_0 e^{at} \qquad (3)$$

How? ① ~~so~~ find a sol'n: $x(t) = e^{at}$

② "general sol'n" of the DE is $x(t) = Ce^{at}$ for any $C$ (linearity)

③ plug general sol'n into IC, solve for $C$:

$$x(0) = Ce^0 \Rightarrow \boxed{C = x_0}$$

Note: $a$ could be complex $a = \alpha + i\beta$ and (3) is still a sol'n:

$$x_0 e^{at} = x_0 e^{(\alpha+i\beta)t} = x_0 e^{\alpha t} e^{i\beta t}$$
$$= x_0 e^{\alpha t} \underbrace{(\cos\beta t + i \sin \beta t)}_{\text{Im}(a)=\beta \text{ gives oscillation}}$$

$\text{Re}(a) = \alpha$ corresponds to:

$\alpha > 0$ exponential growth

$\alpha < 0$ " decay

$\alpha = 0$ neither

Next: look at $\frac{d\vec{x}}{dt} = A\vec{x}$

the ~~vector~~ linear vector ~~ODE~~ IVP.

$A$: $n \times n$ matrix

$\vec{x}$: vector of fcns

$$\vec{x}(0) = \vec{x_0}$$