Math 152 Lecture 23

First-order linear system of differential eqns:

soln is vector $\vec{x}(t)$

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) \quad \text{(for all } t) \qquad (*)$$

$\frac{d}{dt}$ acts "elementwise"

$n \times n$ matrix, constant in time

$(*)$ DE.

IVP (4)

often have an IC: $\vec{x}(0) = \vec{x}_0$

Ex write this system in matrix form (4)

Soln is 3 functions which solve these "coupled" DEs

$$\frac{dx_1(t)}{dt} = x_1(t) + x_3(t) \qquad x_1(0) = 1$$

$$\frac{dx_2}{dt} = x_2 \qquad x_2(0) = 1$$

$$\frac{dx_3}{dt} = x_1 + x_3 \qquad x_3(0) = 0$$

$$\Longrightarrow \frac{d\vec{x}}{dt} = A\vec{x} \quad \text{where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

~~Controll~~ controls coupling b/w eqns.

Idea suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ, consider $\vec{x}(t) = e^{\lambda t}\vec{v}$. This is a sol'n.

check: $LHS = \frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v}$ $RHS = A\vec{x} = A(e^{\lambda t}\vec{v})$

$$= e^{\lambda t}\boxed{(A\vec{v})}$$
$$= e^{\lambda t}\boxed{(\lambda\vec{v})}$$
$$= \lambda e^{\lambda t}\vec{v}$$

yes LHS = RHS.

General soln (of the DE ✱)

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \ldots + c_n e^{\lambda_n t}\vec{v}_n$$

where $\vec{c}$ is a vector of unknown constants. This is constructed by "superposition" of soln's.

IVP find $\vec{c}$ by a linear combination calculation to match the IC.

Ex find the general soln of the system from previous example.

Eigen analysis: $\lambda_1 = 0$ $\vec{v}_1 = [-1 \ 0 \ 1]^T$
$\lambda_2 = 1$ $\vec{v}_2 = [0 \ 1 \ 0]^T$
$\lambda_3 = 2$ $\vec{v}_3 = [1 \ 0 \ 1]^T$

So $\vec{x}(t) = c_1\vec{v}_1 + c_2 e^{t}\vec{v}_2 + c_3 e^{2t}\vec{v}_3$

Ex Solve the IVP in the previous previous example

$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

System: $\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & IC \\ \end{array}$

$$\left[\begin{array}{ccc:c} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{c:c} & -1/2 \\ I & 1 \\ & 1/2 \end{array}\right]$$

the sol'n of the IVP is

$$\vec{x}(t) = -\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 + \frac{1}{2}e^{2t} \\ e^{t} \\ -1/2 + \frac{1}{2}e^{2t} \end{bmatrix}$$

What could go wrong? need a "basis of eigenvectors" so we can determine uniquely $\vec{c}_i$ (i.e., a nondefective matrix).

The IVP has a unique sol'n. We have found 1 sol'n. Thus we have found the unique sol'n.

Ex Solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ w/ $A = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

real matrix — real IC

Eigenanalysis: $\lambda_{1,2} = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$, $\vec{v}_{1,2} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$

General sol'n: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

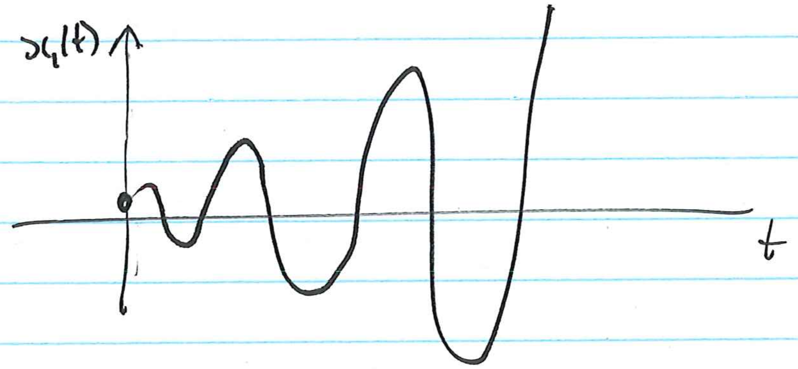
note $e^{\lambda_1 t} = e^{\frac{t}{\sqrt{2}} + i\frac{t}{\sqrt{2}}} = e^{t/\sqrt{2}}\left(\cos t/\sqrt{2} + i \sin t/\sqrt{2}\right)$

$e^{\lambda_2 t} \qquad = \text{''} \left(\text{''} \; - i \; \text{''} \right)$

Note: from this general sol'n, we see

- exponential grow (with "characteristic time" $\sqrt{2}$).
- oscillations with period $\sqrt{2} \cdot 2\pi$

oscillating w/ growing amplitude



Solve for c_1, c_2 by matching IC: $\vec{x}(0) = \vec{x}_0$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{x}_0 \end{array}$$

$$\left[\begin{array}{cc|c} i & -i & 1 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -i/2 \\ 0 & 1 & i/2 \end{array}\right]$$

so $\vec{x}(t) = \underset{c_1}{\frac{-i}{2}} \underset{\vec{v}_1}{\begin{bmatrix} i \\ 1 \end{bmatrix}} \underbrace{e^{t/\sqrt{2}}\left(\cos t/\sqrt{2} + i \sin t/\sqrt{2}\right)}_{e^{\lambda_1 t}}$.

$+ \frac{i}{2} \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{t/\sqrt{2}}\left(\cos \quad - i \sin \quad \right)$

complex conj

This is <u>real</u> but doesn't look like it

$\vec{x}(t) = \vec{U} + i\vec{V} + \vec{U} - i\vec{V}$

$= 2\vec{U} = 2\,\mathrm{Re}\{1^{st} \text{ term}\}$

Often we want the real form of the sol'n.

↳ two approaches

① $\vec{x}(t) = 2\,\mathrm{Re}\{1^{st} \text{ term}\}$

$$\vec{x}(t) = \cancel{2}\ \frac{1}{\cancel{2}}\begin{bmatrix} e^{t/\sqrt{2}} \cos t/\sqrt{2} \\ e^{t/\sqrt{2}} \sin t/\sqrt{2} \end{bmatrix}$$

2. alternatively we look for a different form of the general sol'n.

<u>idea</u> if $\vec{x}(t) = \vec{v}_1 e^{\lambda_1 t}$ is a <u>complex</u> sol'n of $\frac{d\vec{x}}{dt} = A\vec{x}$ when A is <u>real</u>

then ... we have $\lambda_2 = \bar{\lambda}_1$, $\vec{v}_2 = \bar{\vec{v}}_1$

So $\vec{x}(t) = \underbrace{\vec{q}_1}_{\text{real}} + i \underbrace{\vec{q}_2}_{\text{real}}$

and another sol'n is $\vec{y}(t) = \vec{q}_1 - i\vec{q}_2$ $= \vec{v}_2 e^{\lambda_2 t}$

But $\dfrac{\vec{x}(t) + \vec{y}(t)}{2} = \vec{q}_1(t)$ is a sol'n.

and $\dfrac{\vec{x}(t) - \vec{y}(t)}{2i} = \vec{q}_2(t)$ is a sol'n.

So use a general sol'n of:

$$\boxed{\vec{x}(t) = d_1 \vec{q}_1(t) + d_2 \vec{q}_2(t).}$$

which is real. here $\vec{q}_1(t) = \mathrm{Re}(\vec{v}_1 e^{\lambda_1 t})$

$\vec{q}_2(t) = \mathrm{Im}(\vec{v}_1 e^{\lambda_1 t})$

Back to our example:

$$\vec{v}_1 e^{\lambda_1 t} = \underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix} e^{t/\sqrt{2}} \left(\cos \tfrac{t}{\sqrt{2}} + i \sin \tfrac{t}{\sqrt{2}} \right)}$$

let $\vec{q}_1 = \text{Re}\{ \quad \}$

$$= \begin{bmatrix} -e^{t/\sqrt{2}} \sin t/\sqrt{2} \\ e^{t/\sqrt{2}} \cos t/\sqrt{2} \end{bmatrix}$$

$\vec{q}_2 = \text{Im}\{ \quad \}$

$$= \begin{bmatrix} e^{t/\sqrt{2}} \cos t/\sqrt{2} \\ e^{t/\sqrt{2}} \sin t/\sqrt{2} \end{bmatrix}$$

<u>General sol'n:</u> $\vec{x}(t) = d_1 \vec{q}_1(t) + d_2 \vec{q}_2(t)$

IC matching: $\left[\begin{array}{cc|c} \vec{q}_1(0) & \vec{q}_2(0) & \vec{x}_0 \end{array} \right]$

In general

$T\vec{d} = \vec{x}_0$

$\vec{q}$ vectors as columns

$$= \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 0 \end{array} \right] \Rightarrow d_1 = 0, \quad d_2 = 1$$

~GE~...

Sol'n is

$$\vec{x}(t) = \begin{bmatrix} e^{t/\sqrt{2}} \cos t/\sqrt{2} \\ e^{t/\sqrt{2}} \sin t/\sqrt{2} \end{bmatrix}$$

Ex $\dfrac{d\vec{x}}{dt} = A\vec{x}$ $\qquad A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ $\qquad$ find general sol'n. in real form.

Eigenanalysis: $\lambda_1 = 3 \qquad \vec{v}_1 = [1 \ \ 0 \ \ 1]^T$

$\lambda_{2,3} = \pm i \qquad \vec{v}_{2,3} = [\mp 3i, -3 \pm i, 1]^T$

parts of sol'n:

(1) $e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(2) $e^{\lambda_2 t}\vec{v}_2 = e^{it}\begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix}$

Re $\qquad\qquad$ Im

$\vec{q}_1(t) = \begin{bmatrix} 3\sin t \\ -3\cos t - \sin t \\ \cos t \end{bmatrix} \qquad \vec{q}_2(t) = \begin{bmatrix} -3\cos t \\ \cos t - 3\sin t \\ \sin t \end{bmatrix}$

General sol'n:

$$\vec{x}(t) = d_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 3\sin t \\ -3\cos t - \sin t \\ \cos t \end{bmatrix} + d_3 \begin{bmatrix} -3\cos t \\ \cos t - 3\sin t \\ \sin t \end{bmatrix}$$

Ex solve the above system of DE w/ IC $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & -3 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right]}_{\text{columns come from } t=0 \text{ in general sol'n}} \sim \left[\begin{array}{c|c} I & \begin{matrix} 1.3 \\ -0.3 \\ 0.1 \end{matrix} \end{array}\right]$$

So ~~the~~ IVP sol'n is

$$\vec{x}(t) = \begin{bmatrix} 1.3e^{3t} - 0.9\sin t - 0.3\cos t \\ \cos t \\ 1.3e^{3t} - 0.3\cos t + 0.1\sin t \end{bmatrix}$$