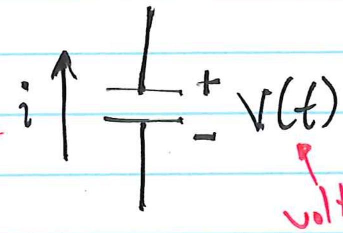
Math 152 - Lecture 24

Return to resistor networks, introduce <u>time dependence</u>

Capacitors



current $i(t)$

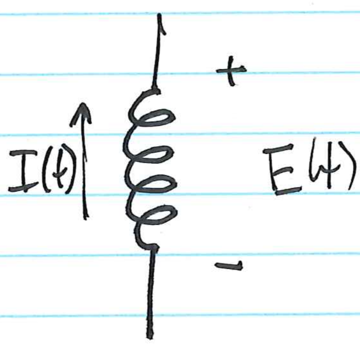
voltage

characterized by $V(t)$ which changes in time in response to the current $i(t)$:

$$\frac{dV}{dt} = -\frac{i}{C}$$

constant, "capacitance" (Farads)

Inductor



current

characterized by $I(t)$ changes according to the voltage $E(t)$

$$\frac{dI}{dt} = -\frac{E}{L}$$

constant "inductance" (Henry's)

At a moment in time t, inductor behaves like a current source $I(t)$ and a capacitor behaves like a voltage source. $V(t)$

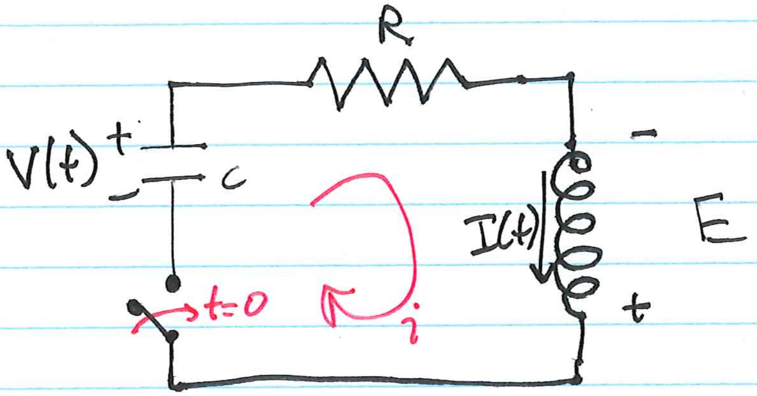
Suppose we have some inductors, capacitors and resistors in network,

Fundamental resistor network problem.

(A) determine the currents (i's) through the capacitors and the ~~voltag~~ voltages (E's) across the inductors in terms of the V's and I's

(B) solve the DEs (with ICs)

Ex



Initially, $V(0)=1$, $I(0)=0$ IC

(A) Fundamental Net. Prob. $iR - E(t) - V(t) = 0$

and $i = I$

$$\Rightarrow \boxed{E = IR - V}$$

(B)

$$\frac{dV}{dt} = \frac{-i}{C} = \frac{-I}{C}$$

$$\frac{dI}{dt} = \frac{-E}{L} = \frac{-(IR-V)}{L} = \frac{V}{L} - \frac{IR}{L}$$

DE

Rewrite as $\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & 1/L \\ -\frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} I \\ V \end{bmatrix}$, $\begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

<u>Eigenanalysis</u>: $\det(A - \lambda I) = 0$

$$\left(-R/L - \lambda\right)(-\lambda) + \frac{1}{CL} = 0 \quad \overset{\cancel{\text{or}}}{\Longrightarrow} \quad \lambda^2 + \frac{R}{L}\lambda + \frac{1}{CL} = 0$$

$$\lambda_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

(at least) two cases:

a) $\frac{1}{CL} < \frac{R^2}{4L^2} \Rightarrow \frac{R^2 C}{4L} > 1$ "large" resistance case

In this case, we have two ~~real~~ real, <u>negative</u>, and <u>distinct</u> eigenvalues

strict ineq

$\lambda_1 = -X + \sqrt{X^2 - \varepsilon}$, $\lambda_2 = \cancel{-X} -2X + \delta$
$= -X + (X - \delta)$
$= -\delta$

<u>Sol'n</u>: $\begin{bmatrix} I \\ V \end{bmatrix}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$ ☆

exponential decay. (as one might expect)

~~[illegible]~~ General sol'n.

b) $\frac{R^2 C}{4L} < 1$ $\lambda_{1,2} = \frac{-R}{2L} \pm ib$, $b = \sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}$

$b \in \mathbb{R}$

exponential decay, characteristic time $\frac{2L}{R}$

sin / cos oscillation with ~~period~~ period $2\pi/b$

For given R, C, L values, we would work out $\vec{v}_1$ and $\vec{v}_2$, determine c_1 and c_2 to match the IC.

c) $\frac{R^2 C}{4L} = 1$ we have one repeated eigenvalue:

$$\lambda = \frac{-R}{2L} \pm 0$$

But it turns out there is no basis of eigenvectors → matrix is defective and of no further concern for Math 152

(Pause for course evaluations — please do one :))

Ex above $R = 2$, $L = 2$, $C = 1$

$$\lambda_{1,2} = -\frac{1}{2} \pm i\frac{1}{2} \qquad \vec{v}_{1,2} = \begin{bmatrix} 1/2 \mp 1/2\, i \\ 1 \end{bmatrix}$$

Recall, we have an alternative form of the general sol'n.

$$\vec{v}_1 e^{\lambda_1 t} = \vec{q}_1(t) + i\,\vec{q}_2(t)$$

$$= e^{-t/2} \begin{bmatrix} 1/2 - 1/2\, i \\ 1 \end{bmatrix} \left(\cos t/2 + i \sin t/2\right)$$

$$= \underbrace{e^{-t/2} \begin{bmatrix} 1/2 \cos t/2 + 1/2 \sin t/2 \\ \cos t/2 \end{bmatrix}}_{\vec{q}_1(t)} + i \underbrace{e^{-t/2} \begin{bmatrix} -1/2 \cos t/2 + 1/2 \sin t/2 \\ \sin t/2 \end{bmatrix}}_{\vec{q}_2(t)}$$

General real sol'n is:

$$\begin{bmatrix} I \\ V \end{bmatrix}(t) = \underset{\mathbb{R}}{d_1}\, \underset{\mathbb{R}^2}{\vec{q}_1(t)} + \underset{\mathbb{R}}{d_2}\, \underset{\mathbb{R}^2}{\vec{q}_2(t)} \qquad ☆☆$$

Sol'n: use IC $= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = d_1 \vec{q}_1(0) + d_2 \vec{q}_2(0)$

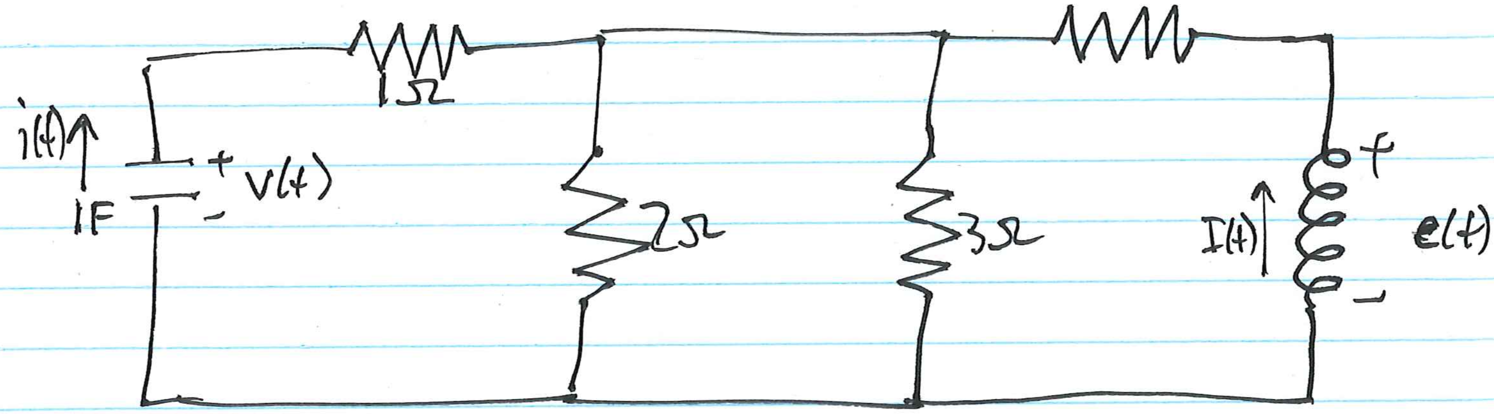
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = d_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

by inspection: $d_1 = 1$ and $d_2 = 1$

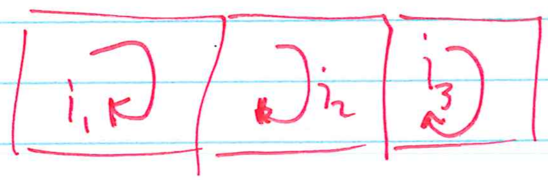
$$\begin{bmatrix} I \\ V \end{bmatrix}(t) = e^{-t/2} \begin{bmatrix} \cancel{1/2 \cos t/2} + 1/2 \sin t/2 - \cancel{1/2 \cos t/2} + 1/2 \sin t/2 \\ \cos t/2 + \sin t/2 \end{bmatrix}$$

plot, etc

Ex $C=1$, $L=10$



We did ~~this~~ the static case before in Lecture 09



$$1 i_1 + 2(i_1 - i_2) - V = 0$$
$$2(i_2 - i_1) + 3(i_2 - i_3) = 0$$
$$5 i_3 + e + 3(i_3 - i_2) = 0$$
$$i_3 = -I$$

called "v_1" in lecture 9

... GE ...

(Fundamental Net Prob)

$$\boxed{\begin{aligned} i = i_1 &= \frac{5}{36} V - \frac{1}{6} I \\ e &= \frac{1}{6} V + 6 I \end{aligned}}$$

These go into the RHS of the DEs:

$$\frac{dV}{dt} = \frac{-i}{C} = \frac{-5}{36} V + \frac{1}{6} I$$

$$\frac{dI}{dt} = \frac{-e}{L} = \frac{-1}{60} V + \frac{6}{10} I$$

time scales

Eigenvalues: $\lambda_1 \approx -0.5939$, $\lambda_2 \approx -0.1450$

$\Rightarrow$ decay, no oscillations, ~~transi~~ decay happens on $\frac{1}{|\lambda_2|} \approx 7s$