Math 152 Lecture ~~23~~ ~~24~~ 25

Exam: soon!!

Office hours: Today, next tues 9am? — watch piazza

Linear systems with matrix unknowns

known $n\times n$ — known $n\times n$ matrix

$$AX = B \qquad (1)$$

unknown $n\times n$

Ex $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$

①
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Find A^{-1}:

$$[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -1/2 \end{bmatrix}$$

$X = A^{-1}B$

$\rightarrow X = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

② **Worse:**

expand

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

to get 4 eqns in 4 unknowns

know now we have $n^2 \times n^2$ matrix

③ slightly Better? recall $A\vec{x} = \vec{b}$

$$G\,[A \,\vdots\, \vec{b}] \sim [I \,\vdots\, \vec{x}]$$

(1) is like n systems of the form $A\vec{x}_j = \vec{b}_j$

columns of X — columns of B

Do GE on all n systems at once:

$$[A \,\vdots\, B] \sim [I \,\vdots\, X]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Linear systems w/ matrix unknowns: transformations

↳ revisit.

Ex $T:\mathbb{R}^2 \to \mathbb{R}^2$ is an unknown linear transformation with

$$T(\vec{a}_1) = \vec{b}_1 \qquad T(\vec{a}_2) = \vec{b}_2$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$$

Find matrix representation of T.

Usual technique: make Π, a 2×2 matrix with columns

$$\Pi = \left[\, T(e_1) \;\vdots\; T(e_2) \,\right] = \left[\, \Pi\vec{i} \;\vdots\; \Pi\vec{j} \,\right] = \left[\, \Pi\begin{pmatrix}1\\0\end{pmatrix} \vdots\; \Pi\begin{pmatrix}0\\1\end{pmatrix} \right]$$

Need to find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$...

Write $\begin{bmatrix}1\\0\end{bmatrix}$ as lin. comb. of $\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$$

T was linear

$$\Rightarrow T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - \frac{1}{2}T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}1\\4\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

1^{st} column of Π

Write $\begin{bmatrix}0\\1\end{bmatrix}$ as lin. comb. of $\vec{a}_1$ and $\vec{a}_2$ ...

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3/2 & 7/2\end{bmatrix}^T$$

$$\Rightarrow \Pi = \begin{bmatrix}-1/2 & 3/2\\ 1/2 & 7/2\end{bmatrix}$$

Note: if $\vec{a}_1$ and $\vec{a}_2$ were complicated,

~~solve:~~ let $A = [\vec{a}_1 \,\vdots\, \vec{a}_2]$

solve $A\vec{c}_1 = \hat{\imath}$ and $A\vec{c}_2 = \hat{\jmath}$

coeff for lin. comb.   coeffs for lin. comb.

why?

$c_{1,1}\vec{a}_1 + c_{1,2}\vec{a}_2 = \vec{e}_1$

$$\underbrace{\begin{bmatrix} \vec{a}_1 & \vdots & \vec{a}_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_{1,1} \\ c_{1,2} \end{bmatrix}}_{\vec{c}_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e}_1 = \hat{\imath}$$

$$AC = [\hat{\imath} \,\vdots\, \hat{\jmath}] = I$$

$$A[\vec{c}_1 \,\vdots\, \vec{c}_2] = \begin{bmatrix} 1 & \vdots & 0 \\ 0 & \vdots & 1 \end{bmatrix}$$

so $C = A^{-1}$ that is $\vec{c}_1$ is 1^{st} col. of A^{-1}

and $\vec{c}_2$ is 2^{nd} col. of A^{-1}

Reminder: solve $AC = I$ for C by doing

$$[A \,\vdots\, I] \sim [I \,\vdots\, \underset{\substack{\uparrow \\ C}}{A^{-1}}]$$

Can we short cut find Π?

recall we know $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ with

unknown.

$$\vec{b}_1 = \Pi \vec{a}_1 \quad \text{and} \quad \vec{b}_2 = \Pi \vec{a}_2$$

known

~~can~~ rewrite as $\left[\vec{b}_1 \,\vdots\, \vec{b}_2\right] = \Pi \left[\vec{a}_1 \,\vdots\, \vec{a}_2\right]$

$$B = \Pi \; A \qquad (2)$$

Ex $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \Pi \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ in our example.

① try: $BA^{-1} = \Pi A A^{-1} = \Pi I = \Pi \Rightarrow \boxed{\Pi = BA^{-1}}$

or ② (2) almost looks like (1), try:

$$\cancel{\#}\ (\Pi A)^T = B^T$$

$$A^T \Pi^T = B^T$$

solve by doing $\left[A^T \,\vdots\, B^T\right] \sim \cancel{\left[I \,\vdots\, (\Pi')^T\right]}$

$$\left[I \,\vdots\, \Pi^T\right]$$

Matrix diagonalization

Suppose $n \times n$ matrix A with

$\lambda_1, \dots, \lambda_n$ eigenvalues

$\vec{v}_1, \dots, \vec{v}_n$ eigenvectors

Say we want a formula for A^m...

Usual technique for finding an unknown "T" (as above) is find $A^m \vec{e}_j$ for each $j = 1, \dots, n$ and use those results as columns (of A^m)

$$\vec{e}_1 = c_{1,1} \vec{v}_1 + c_{1,2} \vec{v}_2 + \dots + c_{1,n} \vec{v}_n$$

(why, b/c $A^m \vec{e}_1 = c_{1,1} \lambda_1^m \vec{v}_1 + c_{1,2} \lambda_2^m \vec{v}_2 + \dots + c_{1,n} \lambda_n^m \vec{v}_n$)

$$\vec{e}_1 = \underbrace{\left[\vec{v}_1 \,\vdots\, \vec{v}_2 \,\vdots\, \dots \,\vdots\, \vec{v}_n \right]}_{V} \begin{bmatrix} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,n} \end{bmatrix} = V \vec{c}_1$$

so $\left[\vec{e}_1 \,\vdots\, \vec{e}_2 \right] = V \left[\vec{c}_1 \; \vec{c}_2 \right] \dots$ $\boxed{I = VC}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} = V \begin{bmatrix} | & | \end{bmatrix} \qquad \Rightarrow C = V^{-1}$$

What about $A^m \vec{e}_j = c_{j,1}\lambda_1^m \vec{v}_1 + c_{j,2}\lambda_2^m \vec{v}_2 + \dots + c_{j,n}\lambda_n^m \vec{v}_n$

$$= V \begin{bmatrix} \lambda_1^m c_{j,1} \\ \lambda_2^m c_{j,2} \\ \vdots \\ \lambda_n^m c_{j,n} \end{bmatrix} = V \underbrace{\begin{bmatrix} \lambda_1^m & & 0 \\ & \lambda_2^m & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}}_{\Lambda^m} \vec{c}_j$$

$$A^m \underbrace{\left[\vec{e}_1 \,\vdots\, \vec{e}_2 \,\vdots\, \dots \,\vdots\, \vec{e}_n\right]}_{I} = V \Lambda^m \overset{V^{-1}}{\cancel{C}}$$

coeff of lin comb of cols.

diag matrix of eigenvalues

$$\Rightarrow \boxed{A^m = V \Lambda^m V^{-1}}$$

in particular:

$$\boxed{A = V \Lambda V^{-1}}$$

matrix diagonalization

cols are eigenvec

diagonal matrix of eigenvals "Lamda"

$AV = V\Lambda \underbrace{\cancel{V^{-1}V}}_{I}$

or $AV = V\Lambda$

$V^{-1}AV = \underbrace{\cancel{V^{-1}V}}_{I}\Lambda$

or $\boxed{V^{-1}AV = \Lambda}$

Now that we have it, we can do:

$$AA = A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda \underbrace{V^{-1}V}_{I} \Lambda V^{-1}$$
$$= V\Lambda I \Lambda V^{-1}$$
$$= V\Lambda\Lambda V^{-1}$$
$$= V\Lambda^2 V^{-1}$$

(in passing)

Early in the course, I mentioned the matrix exponential:

$$\exp(A) = \exp(V\Lambda V^{-1}) = \overset{\text{omit}}{\cdots}$$

$$= V\exp(\Lambda)V^{-1}$$

$$= V\begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}V^{-1}.$$