

**Midterm 2      November 12, 2014      Duration: 50 minutes**

*This test has 5 questions (of unequal value) on 8 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	Total
Points:	10	8	10	7	5	40
Score:						

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Consider the function  $f(x, y) = (x - 1)(y - 2y^2)$ , defined on the rectangle  $R = \{(x, y) \mid 0 \leq x \leq 2, -1 \leq y \leq 1\}$ .

5 marks

- (a) Find and classify all critical points of  $f$  on  $R$ .

Answer:  $(1, 0)$ : saddle, and  $(1, \frac{1}{2})$ : saddle.

**Solution:** Computation gives

$$f_x = y(1 - 2y), \quad f_y = (x - 1)(1 - 4y)$$

and since the first equation has solutions  $y = 0$  and  $y = \frac{1}{2}$ , whereas the second has solutions  $x = 1$  or  $y = \frac{1}{4}$ , the critical points are:  $(1, 0)$  and  $(1, \frac{1}{2})$ . Since

$$f_{xx} = 0, \quad f_{xy} = (1 - 4y),$$

we find that  $D(1, 0) = -1 < 0$  and  $D(1, \frac{1}{2}) = -1 < 0$ , so both are saddle points.

5 marks

- (b) Determine the location and value of the absolute maximum and minimum of  $f$  on  $R$ .

Answer: maximum if  $f(0, -1) = 3$ ,  
minimum is  $f(2, -1) = -3$ .

**Solution:** The absolute maximum and minimum of  $f$  must be on the boundary, which consists of four parts which we call top, bottom, left, right.

Top:  $f(x, 1) = (x - 1)(-1) = 1 - x$  which has maximum  $f(0, 1) = 1$  and minimum  $f(2, 1) = -1$ .

Bottom:  $f(x, -1) = (x - 1)(-3) = 3(1 - x)$  which has maximum  $f(0, -1) = 3$  and minimum  $f(2, -1) = -3$ .

Left:  $f(0, y) = 2y^2 - y$  has derivative  $4y - 1$  so critical point  $y = \frac{1}{4}$  where  $f(0, \frac{1}{4}) = -\frac{1}{8}$ , and at the endpoints  $f(0, -1) = 3$  and  $f(0, 1) = 1$ .

Right:  $f(2, y) = y - 2y^2$ , which is the negative of the left boundary, so the maximum here is  $\frac{1}{8}$  and the minimum is  $-3$ .

Absolute max is  $f(0, -1) = 3$  and absolute minimum is  $f(2, -1) = -3$ .

8 marks

2. A closed cylindrical can must contain 1 litre of water (i.e., 1000 cubic centimetres). Determine the radius  $r$  and height  $h$  of the can that uses the minimal amount of metal.

Answer:  $r = 10(2\pi)^{-1/3}$ ,  $h = 20(2\pi)^{-1/3}$ .

**Solution:** The volume of the can is  $V = \pi r^2 h$  and the amount of metal required is  $S = 2\pi r h + 2\pi r^2$ . We must minimise  $S$  subject to the constraint  $V = 1000$ , so we use the method of Lagrange multipliers.

The equation  $\nabla S = \lambda \nabla V$ , together with the constraint, gives

$$2\pi(h + 2r) = \lambda 2\pi r h \quad \text{or} \quad h + 2r = \lambda r h, \quad (1)$$

$$2\pi r = \lambda \pi r^2, \quad (2)$$

$$\pi r^2 h = 1000. \quad (3)$$

By (2),  $r = 2/\lambda$ , and then from (1) we find that  $h + 4/\lambda = 2h$ , or  $h = 4/\lambda$ . By (3), we have  $16\pi\lambda^{-3} = 1000$ , so  $\lambda = \frac{1}{10}(16\pi)^{1/3}$ .

Finally, we obtain  $r = 2/\lambda = 10(2\pi)^{-1/3}$  and  $h = 4/\lambda = 20(2\pi)^{-1/3}$ .

3. A hiker stands on the slope of a mountain, whose altitude function is given by the equation

$$z = 100 \exp\left(-\frac{x^2 + y^2}{10000} + 0.02\right).$$

Her coordinates are  $(10, 10, 100)$ . She holds a compass: the direction North is given by the vector  $\langle 0, 1 \rangle$ , and East by  $\langle 1, 0 \rangle$ .

3 marks

- (a) The hiker wants to climb to the top in the direction of the steepest slope. In which direction, given by its compass (e.g. North, South, North-East, etc...), must she walk?

Answer: Towards South West.

**Solution:** The steepest slope is given by the direction of the gradient of  $z$ , which is  $\nabla z = \frac{-2z}{10000} \langle x, y \rangle$ . At her current location  $(10, 10, 100)$ ,

$$\nabla z = \frac{-2 \times 100}{10000} \langle 10, 10 \rangle = -\frac{2}{10} \langle 1, 1 \rangle.$$

The direction is thus given by the unit vector  $-\frac{1}{\sqrt{2}} \langle 1, 1 \rangle$ , which corresponds to SW.

4 marks

- (b) She decides that the slope is too high, and that instead she will walk in a direction whose slope is 20%. What direction(s) can she take?

Answer: Towards West or South.

**Solution:** Let  $\mathbf{u}$  be a unit vector giving the direction, and  $\theta$  be the angle between  $\mathbf{u}$  and  $\nabla z$ . The directional derivative  $D_{\mathbf{u}}z$  must be equal to 20%, which translates to:

$$D_{\mathbf{u}}z = |\nabla z| \cos \theta = 0.2.$$

But  $|\nabla z| = \frac{2\sqrt{2}}{10}$ , so  $\theta$  must satisfy  $\cos \theta = \frac{1}{\sqrt{2}}$ . Thus,  $\theta = \pm\pi/4$ . Since the direction of  $\nabla z$  is SW, the direction of  $\mathbf{u}$  is either W or S.

3 marks
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- (c) Another hiker is walking from the point with coordinates  $(10, 10, 100)$  in another direction. A GPS measures that as he walks, the rate of change of his  $x$  coordinate is  $0.2$  m/s, and the rate of change of his  $y$  coordinate is  $0.3$  m/s. What is the rate of change of his altitude  $z$ ?

Answer: $-0.1$ m/s.
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**Solution:** By the chain rule,

$$z'(t) = \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t),$$

therefore the rate of change of  $z$  is

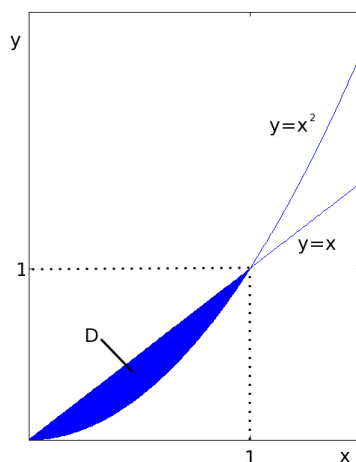
$$-\frac{2}{10}(0.2 + 0.3) = -0.1\text{m/s}.$$

4. Consider the iterated integral

$$I = \int_0^1 \int_y^{\sqrt{y}} \cos(3x^2 - 2x^3) dx dy.$$

2 marks

(a) The integral  $I$  can be regarded as a double integral over a region  $D$  in the  $xy$ -plane. Draw a clearly labeled sketch of the region  $D$ .



**Solution:**

2 marks

(b) Write  $I$  as an iterated integral with the order of integration reversed.

$$\text{Answer: } I = \int_0^1 \int_{x^2}^x \cos(3x^2 - 2x^3) dy dx$$

**Solution:** Reversing the order gives

$$I = \int_0^1 \int_{x^2}^x \cos(3x^2 - 2x^3) dy dx.$$

3 marks

(c) Evaluate the integral  $I$ .Answer:  $\frac{1}{6} \sin 1$ **Solution:**

$$\begin{aligned} I &= \int_0^1 \int_{x^2}^x \cos(3x^2 - 2x^3) dy dx \\ &= \int_0^1 (x - x^2) \cos(3x^2 - 2x^3) dx \\ &= \frac{1}{6} \int_0^1 \cos u du \quad (u = 3x^2 - 2x^3, \quad du = (6x - 6x^2) dx) \\ &= \frac{1}{6} \sin u \Big|_0^1 = \frac{1}{6} \sin 1. \end{aligned}$$

5 marks

5. Consider the region in space consisting of points  $(x, y, z)$  which lie inside the cylinder  $x^2 + y^2 \leq 1$ , above the  $xy$ -plane, and below the plane  $x + y + z = 2$ . Using any method, determine the volume  $V$  of this region. (As always, explain your work; it may be helpful to use the fact that the area of the unit disk is  $\pi$ .)

Answer:  $2\pi$ 

**Solution:** Let  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$  be the unit disk in the  $xy$ -plane. The desired volume  $V$  is

$$\begin{aligned} V &= \iint_D (2 - x - y) dA \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x - y) dy dx \\ &= \int_{-1}^1 \left( [2 - x]y - \frac{1}{2}y^2 \right) \Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx \\ &= \int_{-1}^1 (2 - x)2\sqrt{1 - x^2} dx = \int_{-1}^1 4\sqrt{1 - x^2} dx. \end{aligned}$$

The last integral is 4 times the area of half the disk, so  $4 \times \frac{1}{2}\pi$ , and hence  $V = 2\pi$ .