Midterm 2 November 12, 2014 Duration: 50 minutes

This test has ***5 questions*** *(of unequal value) on* ***8 pages****, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	Total
Points:	10	8	10	7	5	40
Score:						

1. Consider the function $f(x, y) = (x - 1)(y - 2y^2)$, defined on the rectangle $R = \{(x, y) \mid 0 \leq x \leq 2,\ -1 \leq y \leq 1\}$.

5 marks (a) Find and classify all critical points of f on R.

Answer: $(1, 0)$: saddle, and $(1, \frac{1}{2})$: saddle.

Solution: Computation gives

$$f_x = y(1 - 2y), \qquad f_y = (x - 1)(1 - 4y)$$

and since the first equation has solutions $y = 0$ and $y = \frac{1}{2}$, whereas the second has solutions $x = 1$ or $y = \frac{1}{4}$, the critical points are: $(1, 0)$ and $(1, \frac{1}{2})$. Since

$$f_{xx} = 0, \qquad f_{xy} = (1 - 4y),$$

we find that $D(1, 0) = -1 < 0$ and $D(1, \frac{1}{2}) = -1 < 0$, so both are saddle points.

5 marks (b) Determine the location and value of the absolute maximum and minimum of f on R.

Answer: maximum if $f(0, -1) = 3$,
minimum is $f(2, -1) = -3$.

Solution: The absolute maximum and minimum of f must be on the boundary, which consists of four parts which we call top, bottom, left, right.

Top: $f(x, 1) = (x - 1)(-1) = 1 - x$ which has maximum $f(0, 1) = 1$ and minimum $f(2, 1) = -1$.

Bottom: $f(x, -1) = (x - 1)(-3) = 3(1 - x)$ which has maximum $f(0, -1) = 3$ and minimum $f(2, -1) = -3$.

Left: $f(0, y) = 2y^2 - y$ has derivative $4y - 1$ so critical point $y = \frac{1}{4}$ where $f(0, \frac{1}{4}) = -\frac{1}{8}$, and at the endpoints $f(0, -1) = 3$ and $f(0, 1) = 1$.

Right: $f(2, y) = y - 2y^2$, which is the negative of the left boundary, so the maximum here is $\frac{1}{8}$ and the minimum is -3.

Absolute max is $f(0, -1) = 3$ and absolute minimum is $f(2, -1) = -3$.

8 marks 2. A closed cylindrical can must contain 1 litre of water (i.e., 1000 cubic centimetres). Determine the radius r and height h of the can that uses the minimal amount of metal.

Answer: $r = 10(2\pi)^{-1/3}$, $h = 20(2\pi)^{-1/3}$.

Solution: The volume of the can is $V = \pi r^2 h$ and the amount of metal required is $S = 2\pi rh + 2\pi r^2$. We must minimise S subject to the constraint $V = 1000$, so we use the method of Lagrange multipliers.

The equation $\nabla S = \lambda \nabla V$, together with the constraint, gives

$$\begin{aligned} 2\pi(h+2r) &= \lambda 2\pi rh \quad \text{or} \quad h+2r = \lambda rh, && (1)\\ 2\pi r &= \lambda \pi r^2, && (2)\\ \pi r^2 h &= 1000. && (3) \end{aligned}$$

By (2), $r = 2/\lambda$, and then from (1) we find that $h + 4/\lambda = 2h$, or $h = 4/\lambda$. By (3), we have $16\pi\lambda^{-3} = 1000$, so $\lambda = \frac{1}{10}(16\pi)^{1/3}$.

Finally, we obtain $r = 2/\lambda = 10(2\pi)^{-1/3}$ and $h = 4/\lambda = 20(2\pi)^{-1/3}$.

3. A hiker stands on the slope of a mountain, whose altitude function is given by the equation

$$z = 100 \exp\left(-\frac{x^2+y^2}{10000} + 0.02\right).$$

Her coordinates are $(10, 10, 100)$. She holds a compass: the direction North is given by the vector $\langle 0, 1\rangle$, and East by $\langle 1, 0\rangle$.

3 marks

(a) The hiker wants to climb to the top in the direction of the steepest slope. In which direction, given by its compass (e.g. North, South, North-East, etc...), must she walk?

Answer: Towards South West.

Solution: The steepest slope is given by the direction of the gradient of z, which is $\nabla z = \frac{-2z}{10000}\langle x, y\rangle$. At her current location $(10, 10, 100)$,

$$\nabla z = \frac{-2 \times 100}{10000}\langle 10, 10\rangle = -\frac{2}{10}\langle 1, 1\rangle.$$

The direction is thus given by the unit vector $-\frac{1}{\sqrt{2}}\langle 1, 1\rangle$, which corresponds to SW.

4 marks

(b) She decides that the slope is too high, and that instead she will walk in a direction whose slope is 20%. What direction(s) can she take?

Answer: Towards West or South.

Solution: Let $\mathbf{u}$ be a unit vector giving the direction, and θ be the angle between $\mathbf{u}$ and ∇z. The directional derivative $D_u z$ must be equal to 20%, which translates to:

$$D_{\mathbf{u}} z = |\nabla z| \cos\theta = 0.2.$$

But $|\nabla z| = \frac{2\sqrt{2}}{10}$, so θ must satisfy $\cos\theta = \frac{1}{\sqrt{2}}$. Thus, $\theta = \pm\pi/4$. Since the direction of ∇z is SW, the direction of $\mathbf{u}$ is either W or S.

3 marks

(c) Another hiker is walking from the point with coordinates $(10, 10, 100)$ in another direction. A GPS measures that as he walks, the rate of change of his x coordinate is 0.2 m/s, and the rate of change of his y coordinate is 0.3 m/s. What is the rate of change of his altitude z?

Answer: -0.1 m/s.

Solution: By the chain rule,

$$z'(t) = \frac{\partial z}{\partial x} x'(t) + \frac{\partial z}{\partial y} y'(t),$$

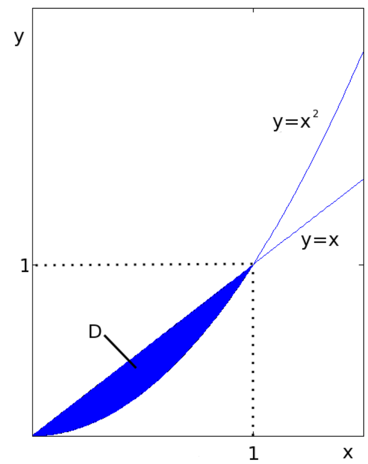
therefore the rate of change of z is

$$-\frac{2}{10}\,(0.2 + 0.3) = -0.1\text{m/s}.$$

4. Consider the iterated integral

$$I = \int_0^1 \int_y^{\sqrt{y}} \cos(3x^2 - 2x^3)\,dx\,dy.$$

$\boxed{\text{2 marks}}$ (a) The integral I can be regarded as a double integral over a region D in the xy-plane. Draw a clearly labeled sketch of the region D.



Solution:

$\boxed{\text{2 marks}}$ (b) Write I as an iterated integral with the order of integration reversed.

Answer: $I = \int_0^1 \int_{x^2}^{x} \cos(3x^2 - 2x^3)\,dy\,dx$

Solution: Reversing the order gives

$$I = \int_0^1 \int_{x^2}^{x} \cos(3x^2 - 2x^3)\,dy\,dx.$$

3 marks

(c) Evaluate the integral I.

Answer: $\frac{1}{6}\sin 1$

Solution:

$$\begin{aligned} I &= \int_0^1 \int_{x^2}^{x} \cos(3x^2 - 2x^3)\,dy\,dx \\ &= \int_0^1 (x - x^2)\cos(3x^2 - 2x^3)\,dx \\ &= \frac{1}{6}\int_0^1 \cos u\,du \qquad \big(u = 3x^2 - 2x^3, \;\; du = (6x - 6x^2)dx\big) \\ &= \frac{1}{6}\sin u\big|_0^1 = \frac{1}{6}\sin 1. \end{aligned}$$

5 marks

5. Consider the region in space consisting of points (x, y, z) which lie inside the cylinder $x^2 + y^2 \leq 1$, above the xy-plane, and below the plane $x + y + z = 2$. Using any method, determine the volume V of this region. (As always, explain your work; it may be helpful to use the fact that the area of the unit disk is π.)

Answer: 2π

Solution: Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ be the unit disk in the xy-plane. The desired volume V is

$$\begin{aligned} V &= \int\int_D (2 - x - y)dA \\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x - y)dydx \\ &= \int_{-1}^{1} ([2 - x]y - \frac{1}{2}y^2)|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} (2 - x)2\sqrt{1 - x^2}dx = \int_{-1}^{1} 4\sqrt{1 - x^2}dx. \end{aligned}$$

The last integral is 4 times the area of half the disk, so $4 \times \frac{1}{2}\pi$, and hence $V = 2\pi$.