

Midterm 2 November 18, 2015 Duration: 50 minutes

This test has 5 questions on 6 pages, each worth 8 points, for a total of 40 points.

Dr. G. Slade, Dr. C. Macdonald, Dr. B. Krause, Dr. M. Murugan

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: _____ Last Name: _____

Student No.: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	Total
Points:	8	8	8	8	8	40
Score:						

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

8 marks

1. Find all critical points of

$$f(x, y) = x^3 + y^3 - 12xy,$$

and classify each critical point as a local maximum, a local minimum, or a saddle point.

Answer: $(0, 0)$ is a saddle point,
 $(4, 4)$ is a local minimum.

Solution: The critical points are the solutions to

$$\begin{aligned}f_x &= 3x^2 - 12y = 0 \\f_y &= 3y^2 - 12x = 0.\end{aligned}$$

From the first equation we obtain $y = x^2/4$. We insert this into the second equation to obtain $x^4 - 64x = x(x^3 - 64) = 0$. Thus $x = 0$ or $x = 4$, and the critical points are $(0, 0)$ and $(4, 4)$.

For their classification, we compute

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -12,$$

and so $D = f_{xx}f_{yy} - f_{xy}^2$ is

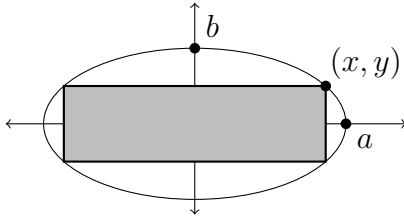
$$D(0, 0) = -144 < 0, \quad D(4, 4) = 36 \cdot 4 \cdot 4 - 144 > 0.$$

Therefore $(0, 0)$ is a saddle point, and since $f_{xx}(4, 4) = 24 > 0$, $(4, 4)$ is a local minimum.

8 marks

2. Using Lagrange multipliers, determine the area of the largest rectangle (with sides parallel to the x and y axes) that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Answer: $2ab$

Solution: The area of the rectangle, whose corner in the first quadrant is (x, y) , is $A = (2x)(2y) = 4xy$. The constraint is $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since $\vec{\nabla}A = \langle 4y, 4x \rangle$ and $\vec{\nabla}g = \langle \frac{2x}{a^2}, \frac{2y}{b^2} \rangle$, we have the three equations

$$\begin{aligned} 4y &= \lambda \frac{2x}{a^2} \\ 4x &= \lambda \frac{2y}{b^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1. \end{aligned}$$

In each of the first two equations, we isolate 2λ and find that $4ya^2/x = 4xb^2/y$, or $y^2 = b^2x^2/a^2$. Thus $y = bx/a$. From the constraint equation, we obtain

$$\frac{x^2}{a^2} + \frac{b^2x^2}{a^2} \frac{1}{b^2} = 1.$$

This gives $2x^2 = a^2$, so $x = a/\sqrt{2}$, and hence $y = bx/a = b/\sqrt{2}$ (we take the positive square roots since (x, y) lies in the first quadrant). This must provide a maximum, since the boundary values $x = 0$ and $x = a$ give $A = 0$. The maximum area is $A = 4(a/\sqrt{2})(b/\sqrt{2}) = 2ab$.

- 2 marks 3. (a) Let $f(x, y) = y^4 \sin x$. Find the gradient $\vec{\nabla} f$.

$$\text{Answer: } \vec{\nabla} f = \langle y^4 \cos x, 4y^3 \sin x \rangle.$$

Solution: $\vec{\nabla} f = \langle f_x, f_y \rangle$; compute the partial derivatives f_x and f_y .

- 2 marks (b) Calculate the directional derivative for f at the point $(\frac{\pi}{2}, 1)$ in the direction given by $\vec{u} = \langle 2, 1 \rangle$.

$$\text{Answer: } \frac{4}{\sqrt{5}}$$

Solution: Since $(\vec{\nabla} f)(\frac{\pi}{2}, 1) = \langle 0, 4 \rangle$ and $|\vec{u}| = \sqrt{5}$,

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 0, 4 \rangle \cdot \langle 2, 1 \rangle \frac{1}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$

- 2 marks (c) Still considering the point $(\frac{\pi}{2}, 1)$, determine the (normalized) direction or directions in which f is changing at rate 2.

$$\text{Answer: } \vec{v} = \langle \pm \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

Solution: The direction is any unit vector \vec{v} such that

$$\vec{\nabla} f \cdot \vec{v} = \langle 0, 4 \rangle \cdot \langle v_1, v_2 \rangle = 2.$$

Thus $v_2 = \frac{1}{2}$, and $v_1^2 + \frac{1}{4} = 1$ gives $v_1 = \pm \frac{\sqrt{3}}{2}$.

- 2 marks (d) What is the maximal rate of change of f at the point $(\frac{\pi}{2}, 1)$, and in what (normalized) direction is this maximal rate of change achieved?

$$\text{Answer: } 4 \text{ in direction } \langle 0, 1 \rangle$$

Solution: The maximal rate of change is $|\vec{\nabla} f| = \sqrt{0^2 + 4^2} = 4$, in direction $\vec{\nabla} f = \langle 0, 4 \rangle$ which is parallel to the unit vector $\langle 0, 1 \rangle$.

4. Consider the integral

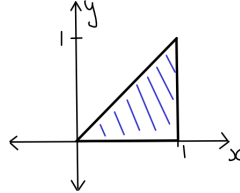
$$\int_0^1 \int_y^1 e^{x^2} dx dy.$$

4 marks

(a) Sketch the integration region and write the integral in reversed order.

$$\text{Answer: } \int_0^1 \int_0^x e^{x^2} dy dx$$

Solution:



$$\int_0^1 \int_y^1 e^{x^2} dx dy = \int_0^1 \int_0^x e^{x^2} dy dx.$$

4 marks

(b) Using the result of part (a), evaluate the integral.

$$\text{Answer: } \frac{1}{2}(e - 1)$$

Solution:

$$\begin{aligned} \int_0^1 \int_0^x e^{x^2} dy dx &= \int_0^1 \left[e^{x^2} y \right]_{y=0}^{y=x} dx \\ &= \int_0^1 x e^{x^2} dx. \end{aligned}$$

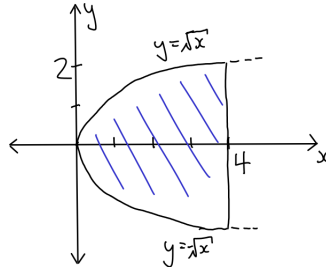
Then we make the substitution $u = x^2$ to evaluate

$$\int_0^1 x e^{x^2} dx = \int_0^1 \frac{1}{2} e^u du = \frac{1}{2} [e^u]_{u=0}^{u=1} = \frac{e - 1}{2}.$$

5. Consider the solid which lies under the surface $z = 1 + x^2y^2$ and above the region in the x - y plane bounded by $x = y^2$ and $x = 4$.

4 marks

- (a) Write down (but don't evaluate) a double integral whose value gives the volume of the solid.

Solution:

$$V = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (1 + x^2y^2) dy dx$$

or

$$V = \int_{-2}^2 \int_{y^2}^4 (1 + x^2y^2) dx dy$$

4 marks

- (b) Evaluate the integral to determine the volume of the solid.

$$\text{Answer: } \frac{2^5}{3} + \frac{2^{11}}{27}$$

Solution:

$$\begin{aligned} V &= \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (1 + x^2y^2) dy dx = 2 \int_0^4 \int_0^{\sqrt{x}} (1 + x^2y^2) dy dx \\ &= 2 \int_0^4 (y + x^2y^3/3) \Big|_{y=0}^{y=\sqrt{x}} dx = 2 \int_0^4 \left(x^{1/2} + \frac{1}{3}x^{7/2} \right) dx \\ &= 2 \left(\frac{2}{3}x^{3/2} + \frac{1}{3 \cdot 9}x^{9/2} \right) \Big|_0^4 = \frac{2^5}{3} + \frac{2^{11}}{27} \end{aligned}$$

or

$$\begin{aligned} V &= \int_{-2}^2 \int_{y^2}^4 (1 + x^2y^2) dx dy = 2 \int_0^2 \int_{y^2}^4 (1 + x^2y^2) dx dy \\ &= 2 \int_0^2 (x + y^2x^3/3) \Big|_{x=y^2}^{x=4} dy = 2 \int_0^2 \left(4 - y^2 + \frac{1}{3}(4^3y^2 - y^8) \right) dy \\ &= 2 \left(4y - \frac{1}{3}y^3 + \frac{1}{3}(4^3y^3/3 - y^9/9) \right) \Big|_0^2 = \frac{2^5}{3} + \frac{2^{11}}{27} \end{aligned}$$