**Midterm 2 November 18, 2015 Duration: 50 minutes**

*This test has* ***5 questions*** *on* ***6 pages****, each worth 8 points, for a total of 40 points.*

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- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: ______________________ Last Name: ______________________

Student No.: ______________________ Section: ______________________

Signature: ____________________________________________

| Question: | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Points: | 8 | 8 | 8 | 8 | 8 | 40 |
| Score: | | | | | | |

8 marks

1. Find all critical points of

$$f(x, y) = x^3 + y^3 - 12xy,$$

and classify each critical point as a local maximum, a local minimum, or a saddle point.

Answer: $(0, 0)$ is a saddle point,
$(4, 4)$ is a local minimum.

**Solution:** The critical points are the solutions to

$$f_x = 3x^2 - 12y = 0$$
$$f_y = 3y^2 - 12x = 0.$$

From the first equation we obtain $y = x^2/4$. We insert this into the second equation to obtain $x^4 - 64x = x(x^3 - 64) = 0$. Thus $x = 0$ or $x = 4$, and the critical points are $(0, 0)$ and $(4, 4)$.

For their classification, we compute

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -12,$$

and so $D = f_{xx}f_{yy} - f_{xy}^2$ is

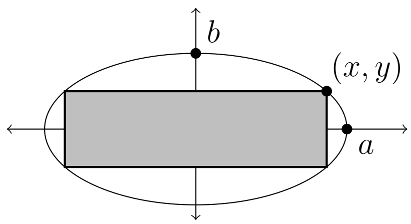
$$D(0, 0) = -144 < 0, \quad D(4, 4) = 36 \cdot 4 \cdot 4 - 144 > 0.$$

Therefore $(0, 0)$ is a saddle point, and since $f_{xx}(4, 4) = 24 > 0$, $(4, 4)$ is a local minimum.

8 marks

2. Using Lagrange multipliers, determine the area of the largest rectangle (with sides parallel to the $x$ and $y$ axes) that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$




Answer: $2ab$

**Solution:** The area of the rectangle, whose corner in the first quadrant is $(x, y)$, is $A = (2x)(2y) = 4xy$. The constraint is $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since $\vec{\nabla} A = \langle 4y, 4x\rangle$ and $\vec{\nabla} g = \langle \frac{2x}{a^2}, \frac{2y}{b^2}\rangle$, we have the three equations

$$\begin{aligned} 4y &= \lambda\frac{2x}{a^2} \\ 4x &= \lambda\frac{2y}{b^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1. \end{aligned}$$

In each of the first two equations, we isolate $2\lambda$ and find that $4ya^2/x = 4xb^2/y$, or $y^2 = b^2x^2/a^2$. Thus $y = bx/a$. From the constraint equation, we obtain

$$\frac{x^2}{a^2} + \frac{b^2x^2}{a^2}\frac{1}{b^2} = 1.$$

This gives $2x^2 = a^2$, so $x = a/\sqrt{2}$, and hence $y = bx/a = b/\sqrt{2}$ (we take the positive square roots since $(x, y)$ lies in the first quadrant). This must provide a maximum, since the boundary values $x = 0$ and $x = a$ give $A = 0$. The maximum area is $A = 4(a/\sqrt{2})(b/\sqrt{2}) = 2ab$.

2 marks 3. (a) Let $f(x, y) = y^4 \sin x$. Find the gradient $\vec{\nabla} f$.

Answer: $\vec{\nabla} f = \langle y^4 \cos x, 4y^3 \sin x \rangle$.

**Solution:** $\vec{\nabla} f = \langle f_x, f_y \rangle$; compute the partial derivatives $f_x$ and $f_y$.

2 marks (b) Calculate the directional derivative for $f$ at the point $(\frac{\pi}{2}, 1)$ in the direction given by $\vec{u} = \langle 2, 1 \rangle$.

Answer: $\frac{4}{\sqrt{5}}$

**Solution:** Since $(\vec{\nabla} f)(\frac{\pi}{2}, 1) = \langle 0, 4 \rangle$ and $|\vec{u}| = \sqrt{5}$,

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 0, 4 \rangle \cdot \langle 2, 1 \rangle \frac{1}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$

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2 marks (c) Still considering the point $(\frac{\pi}{2}, 1)$, determine the (normalized) direction or directions in which $f$ is changing at rate 2.

Answer: $\vec{v} = \langle \pm \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

**Solution:** The direction is any unit vector $\vec{v}$ such that

$$\vec{\nabla} f \cdot \vec{v} = \langle 0, 4 \rangle \cdot \langle v_1, v_2 \rangle = 2.$$

Thus $v_2 = \frac{1}{2}$, and $v_1^2 + \frac{1}{4} = 1$ gives $v_1 = \pm \frac{\sqrt{3}}{2}$.

2 marks (d) What is the maximal rate of change of $f$ at the point $(\frac{\pi}{2}, 1)$, and in what (normalized) direction is this maximal rate of change achieved?

Answer: 4 in direction $\langle 0, 1 \rangle$

**Solution:** The maximal rate of change is $|\vec{\nabla} f| = \sqrt{0^2 + 4^2} = 4$, in direction $\vec{\nabla} f = \langle 0, 4 \rangle$ which is parallel to the unit vector $\langle 0, 1 \rangle$.

4. Consider the integral

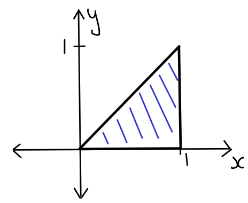
$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy.$$

4 marks

(a) Sketch the integration region and write the integral in reversed order.

Answer: $\int_0^1 \int_0^x e^{x^2}\, dy\, dx$

**Solution:**




$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^x e^{x^2}\, dy\, dx.$$

4 marks

(b) Using the result of part (a), evaluate the integral.

Answer: $\frac{1}{2}(e-1)$

**Solution:**

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{y=x} dx$$
$$= \int_0^1 x e^{x^2}\, dx.$$

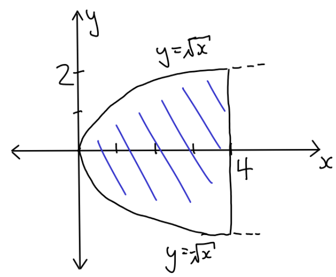
Then we make the substitution $u = x^2$ to evaluate

$$\int_0^1 x e^{x^2}\, dx = \int_0^1 \frac{1}{2} e^u\, du = \frac{1}{2}\left[e^u\right]_{u=0}^{u=1} = \frac{e-1}{2}.$$

5. Consider the solid which lies under the surface $z = 1 + x^2y^2$ and above the region in the $x$-$y$ plane bounded by $x = y^2$ and $x = 4$.

4 marks (a) Write down (but don't evaluate) a double integral whose value gives the volume of the solid.

**Solution:**




$$V = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (1 + x^2y^2)dydx$$

or

$$V = \int_{-2}^2 \int_{y^2}^4 (1 + x^2y^2)dxdy$$

4 marks (b) Evaluate the integral to determine the volume of the solid.

Answer: $\frac{2^5}{3} + \frac{2^{11}}{27}$

**Solution:**

$$\begin{aligned} V &= \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (1 + x^2y^2)dydx = 2\int_0^4 \int_0^{\sqrt{x}} (1 + x^2y^2)dydx \\ &= 2\int_0^4 (y + x^2y^3/3)\Big|_{y=0}^{y=\sqrt{x}} dx = 2\int_0^4 \left(x^{1/2} + \frac{1}{3}x^{7/2}\right) dx \\ &= 2\left(\frac{2}{3}x^{3/2} + \frac{1}{3}\frac{2}{9}x^{9/2}\right)\Big|_0^4 = \frac{2^5}{3} + \frac{2^{11}}{27} \end{aligned}$$

or

$$\begin{aligned} V &= \int_{-2}^2 \int_{y^2}^4 (1 + x^2y^2)dxdy = 2\int_0^2 \int_{y^2}^4 (1 + x^2y^2)dxdy \\ &= 2\int_0^2 (x + y^2x^3/3)\Big|_{x=y^2}^{x=4} dy = 2\int_0^2 \left(4 - y^2 + \frac{1}{3}(4^3y^2 - y^8)\right) dy \\ &= 2\left(4y - \frac{1}{3}y^3 + \frac{1}{3}(4^3y^3/3 - y^9/9)\right)\Big|_0^2 = \frac{2^5}{3} + \frac{2^{11}}{27} \end{aligned}$$