Math 253 Notes on Moments of Inertia

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Colin B. Macdonald, Supplementary notes for Math 253, to follow Section 13.4 "Center of Mass" of our text APEX Calculus 3, version 3.0.

1 Moments of Inertia

We've previously seen *moments* when calculating centre of mass of a lamina. This involved two double integrals:

$$M_y = \int\int_D x\rho(x,y)dA$$
$$M_x = \int\int_D y\rho(x,y)dA$$

These can also be called the "first moments"; here we look at the "second moments" or "moments of inertia".

1.1 Kinetic Energy of a spinning lamina

Suppose our lamina (which lies in the x-y plane) is rotating around the z-axis (note this is orthogonal to the lamina) at a constant angular rotational speed ω radians/s. (E.g., 60 rpm = 1 rev/s = 2π rad/s). Find the *Kinetic Energy* of the lamina. [Draw diagram!]

Riemann sum idea: as before, we consider a small rectangular piece R_{ij} with area $\Delta x\Delta y$. The kinetic energy of a point mass is $\frac{1}{2}mv^2$. Its going to be small in the limit so we use this to get:

$$\frac{1}{2}\rho(x_i, y_j)\Delta x\Delta y|\vec{v}_{ij}|^2.$$

The piece R_{ij} moves faster the further it is from the axis of rotation (z-axis, $(x, y) = (0, 0)$). Different pieces move at different speeds. Our piece has kinetic energy:

$$\frac{1}{2}\rho(x_i, y_j)\Delta x\Delta y\omega^2\left(x_i^2 + y_j^2)\right).$$

So take the Riemann sum over all pieces of the lamina and we get:

$$K = \frac{1}{2}\omega^2\int\int_D(x^2 + y^2)\rho(x, y)dA.$$

We define I_0 the **moment of inertia** about the z-axis as just the integral part:

$$I_0 = \int\int_D(x^2 + y^2)\rho(x, y)dA.$$

Larger I_0 means more energy (work) to rotate the lamina about the z-axis.

1.2 About some other axis?

A similar argument shows how to compute the moment of inertia about some other axis parallel to the z-axis, centred at $(x, y) = (a, b)$:

$$I_0 = \int\int_D ((x-a)^2 + (y-b)^2)\rho(x, y)dA.$$

And in particular about the centre of mass $(x, y) = (\bar{x}, \bar{y})$, this would be:

$$I_{0,c} =$$

1.3 x and y axes of rotation

What about rotating around the x-axis and y-axis? This gives the moment of inertia about the y-axis denoted I_y and the moment of inertia about the x-axis denoted I_x. [Draw diagrams]

$$I_y =$$

$$I_x =$$

Note relationship to previous, $I_0 = $

1.4 Changing the axis of rotation

Suppose we have $I_{0,c}$ and want rotation around z-axis? Let M be overall mass of lamina. We get:

$$I_0 =$$

1.5 Examples

1. Find moment of inertia about the z-axis of a uniform circular disc of radius R and total mass M, centred at the origin.
2. Find same, but with disc centred at point (a, b).
3. Find same, for a uniform rectangular plate, mass M, axis through centre, size $a \times b$.