Math 253 Notes on Partial Differential Equations

September 28, 2016

 Supplementary notes for Math 253, to follow Section 12.3 "Partial Derivatives" of our text APEX Calculus 3, version 3.0.

1 Partial Differential Equations

A partial differential equation (or PDE for short) is an equation, given in terms of partial derivatives, for which the solution is a *function*.

1.1 Laplace's equation

For example, Laplace's equation is:

$$u_{xx} + u_{yy} = 0,$$

or in our other notation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A solution to this will be a *function* $u(x, y)$ whose second partial deriviative in x is the same as the negative of its second partial derivative in y.

Its not easy to just think up such a function! Maybe $u(x, y) = xy$ would work?

In fact, it does, because both second partial are 0 and thus sum to zero. In general, solving PDEs is *hard*, the subject of at least an entire courses and many entire careers. But we've just seen a much easier problem: **checking if a *given* function solves a PDE**.

Solutions of Laplace's equation are called *harmonic functions* and with a name like that they'd better be important and beautiful. And they are. Applications include electrostatics, gravitational potentials, and steady-state heat distribution. They are fundamental in the mathematical theory of complex numbers and other areas.

This document was written as a interactive Jupyter worksheet; this allows me to have the computer do the calculations for me, using Python. You can too at http://pims.jupyter.ca. We do need some bookkeeping stuff, so please excuse the brief interlude.

```
In [39]: from sympy import *
         init_printing()
         x, y, t, a, b, n = symbols('x,y,t,a,b,n')
```

With that out of the way, here's another PDE:

1.2 The Wave Equation

In its simplest form, the wave equation models vibrations in a string by the PDE

$$u_{tt} = a^2 u_{xx}.$$

This model can be derived from Newton's second law, taking u as the lateral (vertical) displacement and by assuming a string of constant density and tension, neglecting friction and gravity and assuming small displacements. These assumptions are reasonable for guitar or violin strings for example.

Now let n be an integer. Is the following expression for $u(x, t)$ a solution of the wave equation for $x \in [0, \pi]$?

```
In [18]: u = cos(a*n*t)*sin(n*x)
         u
```

Out[18]:

$$\sin(nx)\cos(ant)$$

We want to take some partial derivatives of this. We can have the computer do the calculations for us. For example, the partial derivative of u with respect to x is:

```
In [19]: u_x = u.diff(x)
         u_x
```

Out[19]:

$$n\cos(nx)\cos(ant)$$

Similarly, we find

```
In [20]: u_xx = u.diff(x, x)
         u_xx
```

Out[20]:

$$-n^2 \sin(nx)\cos(ant)$$

```
In [21]: u_tt = u.diff(t, t)
         u_tt
```

Out[21]:

$$-a^2 n^2 \sin(nx)\cos(ant)$$

Now let's substitute those expressions into the PDE. If we do this as "LHS - RHS" and get zero, u must be a solution:

```
In [22]: u_tt - a**2*u_xx
```

Out[22]:

$$0$$

Practice exercise: Sketch the above solution for $n = 1$ and $n = 2$, both when $t = 0$. The solution oscillates in time: can you make an animation that shows this? Thinking of a guitar string, what do you suppose the oscillation for $n = 1$ corresponds to musically?

Practice exercise: Show that $u(x, t) = e^{x+at}$ is a solution of the PDE $u_{tt} - a^2 u_{xx} = 0$.

Practice exercise: Show that $u(x, t) = f(x - at)$ is a solution of the PDE $u_{tt} = a^2 u_{xx}$ for *any* function f. Hint: you might not think so at first glance but f is a function of just one variable, say $f(s)$. Hint: think about the chain rule.

1.3 The Heat Equation

Diffusion is a physically-important process commonly modelled by the PDE

$$u_t = u_{xx},$$

whose solution $u(x, t)$ gives the amount of something (like heat) at a point x and time t.

The heat equation can also work in higher-dimensions such as 3D space + time:

$$u_t = u_{xx} + u_{yy} + u_{zz},$$

where solutions are now $u(x, y, z, t)$.

Practice exercise: $u(x, t) = e^{at} \sin(3x)$ is a solution of the heat equation. Determine the value or values of the constant a. If you're doing this in Jupyter, you could start with:

```
In [33]: u = exp(a*t)*sin(3*x)
         u_xx = u.diff(x, x)
         u_t = u.diff(t)
```

Practice exercise: Show that if $u(x, t)$ and $v(x, t)$ are solutions to the heat equation, then so is $au + bv$ where a, b are constants. This means the heat equation is a *linear PDE*.

1.4 Nonlinear problems

Linear PDEs are broadly responsible for advances in Physics from the late 1800s until sometime in last century. But nonlinear problems are even more exciting. They are typically not solvable in the sense of finding a formula for "the answer". Instead, vast computational resources are spend finding approximate numerical solutions via large linear algebra problems. Examples include weather prediction, fluid flow problems ("CFD"), climate modelling, and cosmic simulation. Its not impossible that we are living inside someone else's large-scale PDE simulation.

Here's one example: the Gray-Scott model is two heat equation PDEs coupled together:

$$u_t = D_u (u_{xx} + u_{yy}) - uv^2 + F(1 - u),$$

$$v_t = D_v (v_{xx} + v_{yy}) + uv^2 - (F + k)v,$$

where D_u, D_v, F, k are constants. This thing can produce some gnarly biological patterns: mrob.com/pub/comp/xmorphia. To quote Alan Turing (who started this, 40 years ahead of the nonlinearity mathbio renaissance) "waves on cows and waves on leopards".