The Chain Rule §12.5

HW3: online this afternoon.
↳ attempt Q1 and Q3 before Wed!

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single variable calc: $f(g(t)) = y(t)$

$$\Rightarrow \quad y'(t) = f'(g(t))\, g'(t)$$

Alternatively $y = f(x)$ with $x = g(t)$

then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

same as $\dfrac{df}{dx}$

Suppose we have a path in the x-y plane $x = g(t)$ and $y = h(t)$. Say surface $z = f(x,y)$. Along the path $\underset{z(t)}{z} = f(g(t), h(t))$

APEX thm 107: "chain rule part 1"

$$\boxed{\frac{dz}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

evaluated at $(g(t), h(t))$

Alternatively: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

Caution!

Mnemonic : " $\dfrac{\text{total differential}}{dt}$ "

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

can be made precise using the $\Delta z = dz + \varepsilon_1 dx + \varepsilon_2 dy$ idea from previous lecture

theory

(not examinable)

I made a mess of this, go thru it carefully!

Ex (APEX ex 416, see also)

$$w = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$x = \sin t \Rightarrow \dfrac{dx}{dt} = \cos t$

$y = \cos t \Rightarrow \dfrac{dy}{dt} = -\sin t$

$z = \tan t \Rightarrow \dfrac{dz}{dt} = \tan^2 t + 1$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \cancel{\left(\frac{\sin t}{\sec^2 t}\right)\cos t + \left(\frac{\cos t}{\sec^2 t}\right)(-\sin t)} + \left(\frac{\tan t}{\cancel{\sec^2 t}}\right)\left(\cancel{\sec^2 t}\right)$$

$$= \tan t$$

$$\frac{\partial w}{\partial x} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2+z^2}\cdot\cancel{2}x = \frac{x}{x^2+y^2+z^2}$$

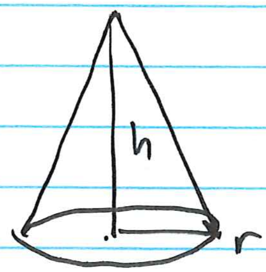
$$\frac{\partial w}{\partial y} = \frac{y}{x^2+y^2+z^2} \qquad \frac{\partial w}{\partial z} = \frac{z}{\text{"}}$$

Note: $x^2+y^2+z^2 = \underbrace{\sin^2 t + \cos^2 t}_{1} + \tan^2 t = \sec^2 t$

Could also sub in $x(t)$, $y(t)$, $z(t)$ and do directly

1. $w(t) = \ln\sqrt{\sin^2 t + \cos^2 t + \dots}$
2. ask 1st-year student

Ex (see also APEX ex 4.17)



Cone, radius is increasing at 1.8 cm/s and the height is decreasing at −2.5 cm/s

What is the rate of change of volume when r = 120 cm and h = 140 cm?

$$V = \frac{1}{3}\pi r^2 h, \quad \text{want } \frac{dV}{dt} =$$

(Note cannot sub r(t) and h(t) in)

$$\boxed{\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}}$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h$$

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

$$\frac{dV}{dt} = \frac{2\pi}{3} r h \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$= \frac{2\pi}{3} \times 120 \times 140 \times 1.8 + \frac{1}{3}\pi (120)^2 \times (-2.5)$$

$$\approx 25635 \text{ cm}^3/\text{s} \approx 25 \text{ L/s}$$

Chain Rule, Part 2

Suppose $z = f(x,y)$ with $x = g(s,t)$, $y = h(s,t)$

e.g. change of variables cartesion to polar

$$\boxed{\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$

Matrix

(Check!!)

$$\underbrace{\begin{bmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial s} \end{bmatrix}}_{\text{vec}} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}}_{\text{vec}}$$