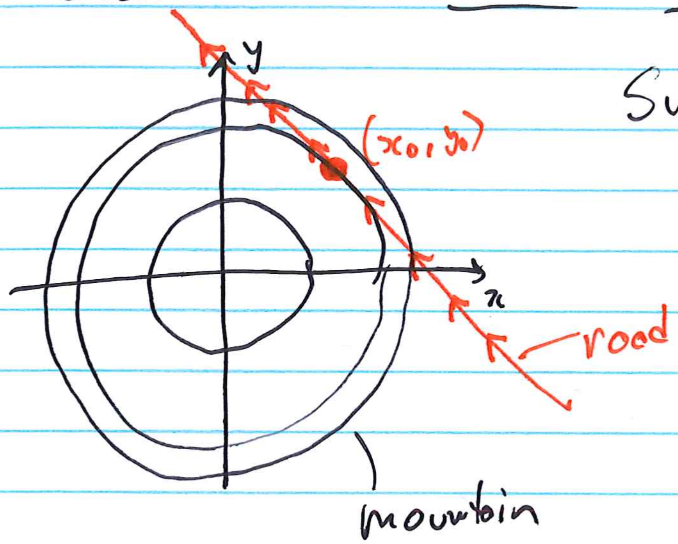
§12.6 The directional derivative and the gradient



Suppose we have surface $z = f(x,y)$

As we drive along a straight road, what is our rate of change of height? at (x_0, y_0)?

Road: $x(t) = x_0 + at$
$y(t) = y_0 + bt$ } $t=0$ is (x_0, y_0)

thru (x_0, y_0)
direction $\vec{u} = \langle a, b \rangle$

$$z(t) = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t)$$ Chain rule!

$$\left.\frac{dz}{dt}\right|_{t=0} = \left. \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b \right|_{t=0}$$

$$\frac{dz}{dt}(0) = a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$$

Notation: $(D_{\vec{u}}f)(x_0,y_0)$ is the rate of change of f at a pt (x_0,y_0) in the direction $\underbrace{\vec{u}, \text{ with } \|\vec{u}\|=1.}_{}$

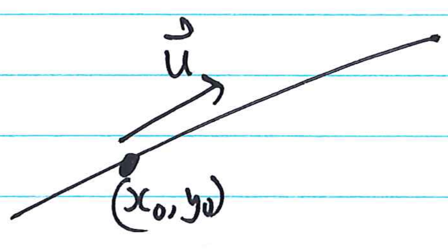
direction ~~vect~~ unit vector.

Def'n: [APEX def'n 90] let $\vec{u} = \langle a,b \rangle$, $a^2+b^2=1$

$$(D_{\vec{u}}f)(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

Derivation (Thm 110 APEX) Let $z=f(x,y)$, suppose $\vec{u}=\langle a,b\rangle$ is a unit vector and consider pt (x_0,y_0).

let $x(t) = x_0 + at$, $y(t) = y_0 + bt$.



Then $(D_{\vec{u}}f)(x_0,y_0) = \frac{df}{dt}(0)$

(where $f(t) := f(x(t), y(t))$)

$= \dots = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \langle a,b\rangle$

as above

$= \langle f_x, f_y\rangle\big|_{x_0,y_0} \cdot \vec{u}$

Result $\boxed{D_{\vec{u}}f = \langle f_x, f_y\rangle \cdot \vec{u}}$ ← "The directional derivative".

<u>Def'n</u> The <u>gradient</u> of $f(x,y)$ is

$$\vec{\text{grad}}\, f = \nabla f = \langle f_x, f_y \rangle = \vec{\nabla} f$$

Note · $f(x,y)$ is a scalar fcn of 2 vars
· ∇f is a <u>vector fcn</u> of 2 vars.

<u>Note</u> $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$ $\quad (\|\vec{u}\| = 1)$

"nabla" or "del"

<u>careful</u>, you must normalize!
"compute directional derivative of ... in the direction $\langle 3,4 \rangle$

Ex let $f(x,y) = x^2+y^2$, $\begin{cases} \vec{u}_1 = \frac{\langle 1, 1 \rangle}{\sqrt{2}} \\ \vec{u}_2 = \frac{\langle 1, -1 \rangle}{\sqrt{2}} \\ \vec{u}_3 = \langle 1, 0 \rangle = \vec{i} \end{cases}$

Compute $D_{\vec{u}}f$ for each direction at the point $(1, 1)$.

$$\vec{\nabla} f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\nabla f|_{(1,1)}$

① $D_{\vec{u}_1} f = \langle 2, 2 \rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}} = \frac{2+2}{\sqrt{2}} = 4/\sqrt{2}$

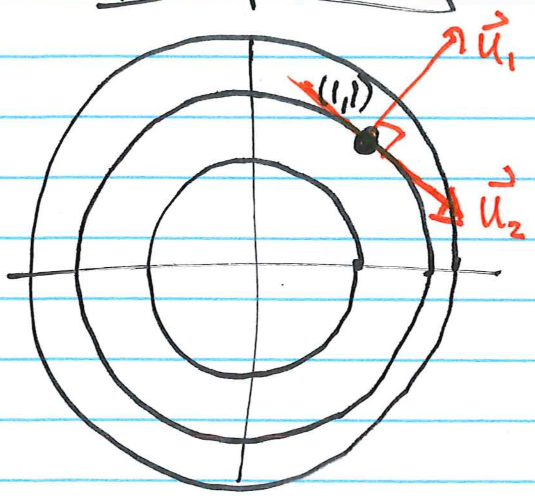
② $D_{\vec{u}_2} f = \langle 2, 2 \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \frac{2-2}{\sqrt{2}} = 0$

③ $D_{\vec{u}_3} f = \langle 2, 2 \rangle \cdot \langle 1, 0 \rangle = \cancel{f_x} \, 2 \; (= f_x(1,1))$

$D_{\vec{i}} f = f_x$ and $D_{\vec{j}} f = f_y$

From the gradient, we can recover rate of change in any direction.

Interpretation



① In general, $D_{\vec{u}} f$ is zero whenever $\vec{u}$ is tangential to the contour thru (x_0, y_0) b/c f is constant along contours.

(note: instantaneous rate of change).

(prove by constructing $x(t) = \ldots$ and $y(t) = \ldots$ for tangent line of the $f(x,y) = k$ contour and chain rule 1 as before)

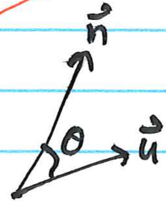
② What is the maximum value of $D_{\vec{u}} f$ over all possible directions?

↳ direction should be $\perp$ to the contour

(intuition)

Consider $\vec{n} = \dfrac{\nabla f}{\|\nabla f\|}$

next day, let's start with just ∇f.



Contrast

($\vec{n}$ and $\vec{u}$ form angle θ)

$$D_{\vec{u}} f = \|\nabla f\| \underbrace{\|\vec{u}\|}_{=1} \cos\theta$$

maximal when $\theta = 0$ (or $\theta = 180$)