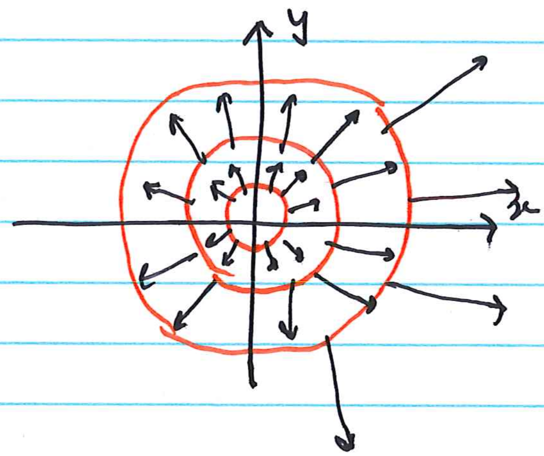
Last day: we started looking at the gradient of $f(x,y)$

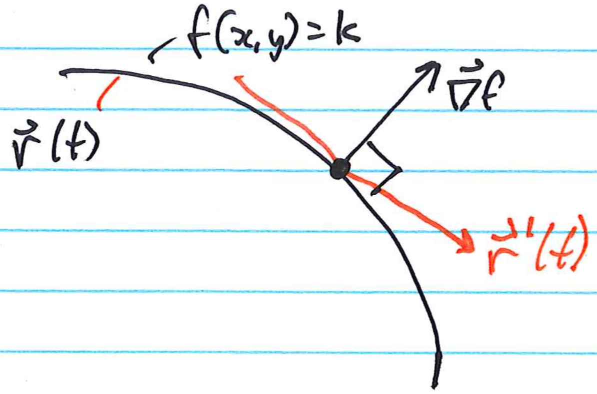
- Gradient of $f(x,y)$ is $\vec{\nabla} f = \langle f_x, f_y \rangle = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$
- $\vec{\nabla} f$ is a <u>vector</u> <u>field</u>: every point in the ~~x-y plane~~ domain of f has a vector



Eg., $f(x,y) = x^2 + y^2$

$\nabla f = \langle 2x, 2y \rangle$

The gradient is orthogonal to the ~~contou~~ contour curves.



<u>Proof</u>: Contour solves $f(x,y) = k$ and it is a curve so $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Note $\vec{r}'(t)$ is tangent to the contour.

$$\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} k = 0$$

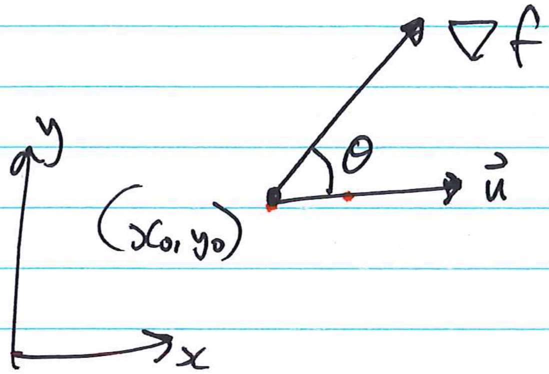
$$\Rightarrow f_x \, x'(t) + f_y \, y'(t) = 0$$

$$\Rightarrow \vec{\nabla} f \cdot \vec{r}'(t) = 0$$

$\Rightarrow \vec{\nabla} f$ is $\perp$ to the tangent of contour curve.

$\Rightarrow \vec{\nabla} f$ is $\perp$ to the contour.

Suppose we're at point (x_0, y_0), what unit vector $\vec{u}$ should we choose to maximize $D_{\vec{u}} f$?



$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

We have $D_{\vec{u}} f = \underbrace{\|\vec{\nabla} f\| \, \|\vec{u}\|}_{1} \underbrace{\cos\theta}$ (*)

can change this angle.

maximal when $\cos = 1$

② $\boxed{D_{\vec{u}} f \text{ is \underline{maximal} when } \vec{u} \text{ is in the same direction as } \vec{\nabla} f}$

① (last day) ~~[illegible]~~ $\cos\theta = 0$, $D_{\vec{u}} f = 0$ ~~along~~ if $\vec{u}$ is along the contour.

Note to do this, take $\vec{u} = \dfrac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$ to maximize $D_{\vec{u}} f$.

③ The <u>maximum</u> <u>value</u> of $D_{\vec{u}} f$ is $\|\vec{\nabla} f\|$ (from (*))

④ The minimum ~~[illegible]~~ (most negative) value of $D_{\vec{u}} f$ is $-\|\vec{\nabla} f\|$ and occurs when $\vec{u} = -\dfrac{\nabla f}{\|\nabla f\|}$.

Gradient & Directional Deriv in functions of (x, y, z)
(§12.6, §12.7)

$F(x,y,z)$, $\quad \vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ 3-vector

$D_{\vec{u}} F = \vec{\nabla} F \cdot \vec{u}$ $\quad \vec{u} = \langle a, b, c \rangle$ w/ $\|\vec{u}\| = 1$

(scalar)

Recall

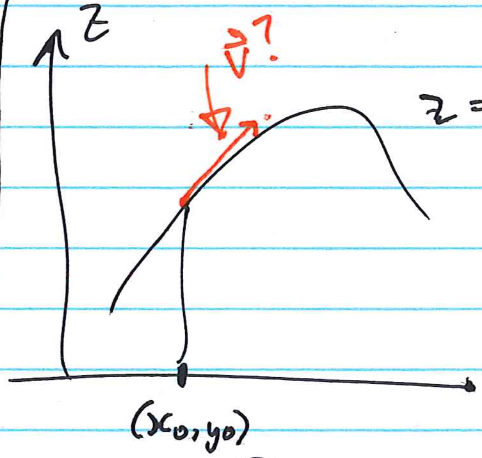
Level surfaces of F are surfaces given implicitly by $F(x,y,z) = k$

$$\boxed{\vec{\nabla} F \text{ (at } (x_0, y_0, z_0)) \text{ orthogonal to the level surface passing through } (x_0, y_0, z_0).}$$

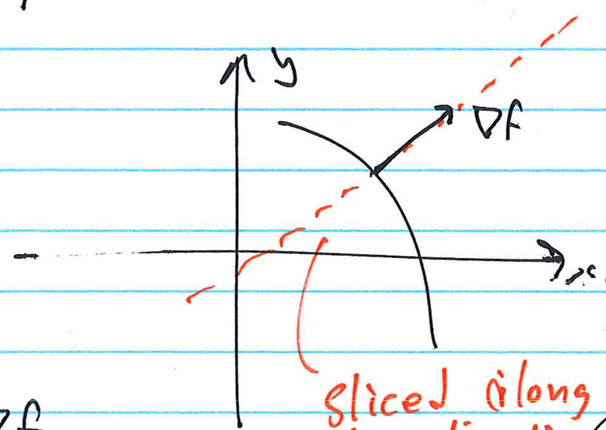
$\Rightarrow$ $\vec{\nabla} F$ is "a normal vector to" the level surface $F(x,y,z) = k$.

↳ "Easy" to get tangent plane at a point (x_0, y_0, z_0) on a implicit surface.

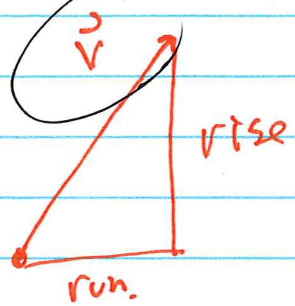
For surface $z = f(x,y)$, gradient points (in x-y plane) ! toward the path of steepest ascent, $\perp$ to the contours.



$\rightarrow \vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$



sliced along the direction $\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$



~~What~~ What is $D_{\vec{u}} f = \|\nabla f\|$

rate of change: interprete as $\approx \dfrac{\text{rise}}{\text{run}}$

Slope is $\dfrac{\|\nabla f\|}{1}$

What about $\vec{v}$?

$$\vec{V} = \overset{u_1}{u(1)}\,\hat{\imath} + \overset{u_2}{u(2)}\,\hat{\jmath} + b\,\hat{k}$$

$$= \overset{u_1}{u(1)}\,\hat{\imath} + \overset{u_2}{u(2)}\,\hat{\jmath} + \|\nabla f\|\,\hat{k}$$

(think about for webwork).

consider this ill-advised diversion as FYI for Webwork but it shouldn't be part of the lecture.