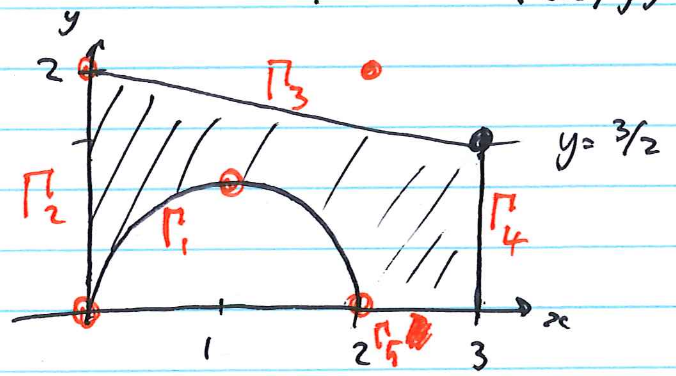
Last day : example of min/max on bounded domain. $f(x,y) = \dots$ (last day).

Today!



Step 1 : find c.p. and classify

Step 2 : solve 1-var calculus prob on the boundary.

Γ are curves in x-y plane

last day Γ_2 , Γ_5 : $f \equiv 0$

Γ_3 : parameterize: $\langle x(t), y(t) \rangle$, $t \in [0,b]$
$= \langle t, -\frac{1}{6}t + 2 \rangle$, $t \in [0,3]$

alt: $\boxed{\Gamma_3 : \langle x, -\frac{1}{6}x + 2 \rangle , \quad x \in [0,3]}$

$g_3(x) = f(x, -\frac{1}{6}x + 2) =$ find a 101 student.

Γ_1 : $\langle x(t), y(t) \rangle = \langle 1 + \cos t, \sin t \rangle$ $t \in [0, \pi]$

alt: $(x-1)^2 + y^2 = 1$ $y = \pm\sqrt{1-(x-1)^2}$ only + $x \in [0,2]$

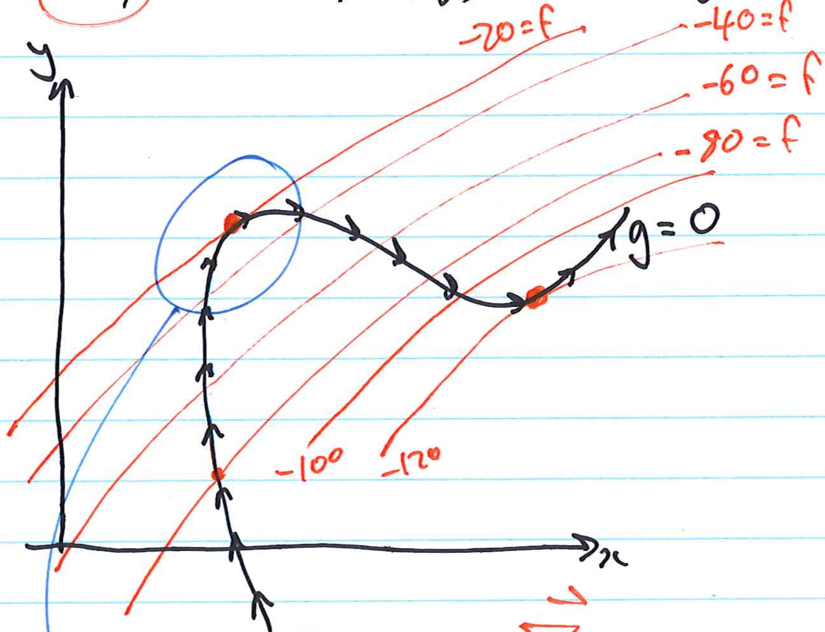
$g_1(t) = f(1+\cos t, \sin t) = \dots$

not in APEX, see website.

Constrained Optimization: Lagrange Multipliers

We've seen min/max $f(x,y)$ with $x, y \in$ domain (eg. previous problem)

Another class of problems looks like max/min $f(x,y)$ subject to $\underbrace{g(x,y) = 0}_{\text{algebraic constraint}}$



We walk along the curve $g=0$, measuring f at each step, find the largest value for f.

normal vector of $g=0$ is // to normal vector of $f=k$ $\vec{\nabla} f$

$$\Rightarrow \quad \vec{\nabla} f = \lambda \vec{\nabla} g$$

λ ← Lagrange Multiplier.

3 eqns and 3 unknowns

nonlinear

$$\left.\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= 0 \end{aligned}\right\} \quad (x, y, \lambda)$$

Alternative derivation : given f and g.

$$\mathcal{L}(x,y,\lambda) := f(x,y) - \lambda g(x,y)$$

new fcn

$\vec{\nabla}\mathcal{L} = 0$ to find c.p. of $\mathcal{L}$.

w.r.t. (x,y,λ)

$$\left.\begin{aligned} \mathcal{L}_x &= f_x - \lambda g_x = 0 \\ \mathcal{L}_y &= f_y - \lambda g_y = 0 \\ \mathcal{L}_\lambda &= 0 - g = 0 \end{aligned}\right\}$$

Same as before.

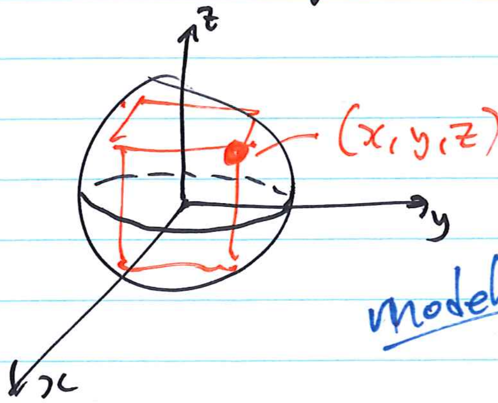
(usually a saddle point of $\mathcal{L}$).

Also works for more variables:

max $F(x,y,z)$ ~~s.t.~~ subject to $G(x,y,z)=0$

$$\Rightarrow \left.\begin{aligned} \vec{\nabla}F &= \lambda \vec{\nabla}G \\ G &= 0 \quad \cancel{g = 0} \end{aligned}\right\}$$ 4 eqns in 4 vars.

Example find the largest box by volume that can be inscribed by a sphere of radius r.



modelling

Volume: $V(x,y,z) = (2x)(2y)(2z)$
$= 8xyz$

Constraint: $\underbrace{x^2+y^2+z^2-r^2}_{G(x,y,z)} = 0$

$$\vec{\nabla} V = \langle 8yz, 8xz, 8xy \rangle$$

$$\vec{\nabla} G = \langle 2x, 2y, 2z \rangle$$

Introduce λ, Lagrange multiplier:

$$\left.\begin{aligned} 8yz &= \lambda 2x \\ 8xz &= \lambda 2y \\ 8xy &= \lambda 2z \\ x^2+y^2+z^2-r^2 &= 0 \end{aligned}\right\}$$

solve for λ

$$\frac{\lambda}{4} = \overbrace{\frac{yz}{x} = \frac{xz}{y}} = \underbrace{\frac{xy}{z}}$$