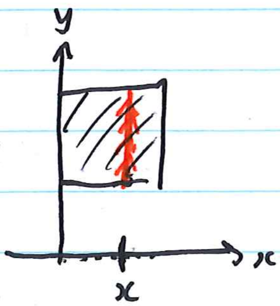
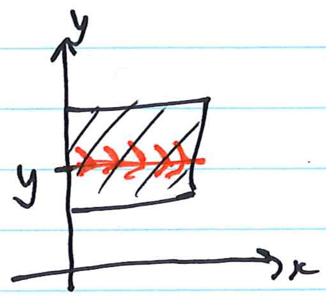
Double integrals §13.1.

Example: evaluate $\iint_R \underbrace{(1-x)y}_{f(x,y)}\, \underbrace{dA}_{dx\,dy}$ $\quad R=[0,1]\times[1,2]$



Sol'n 1

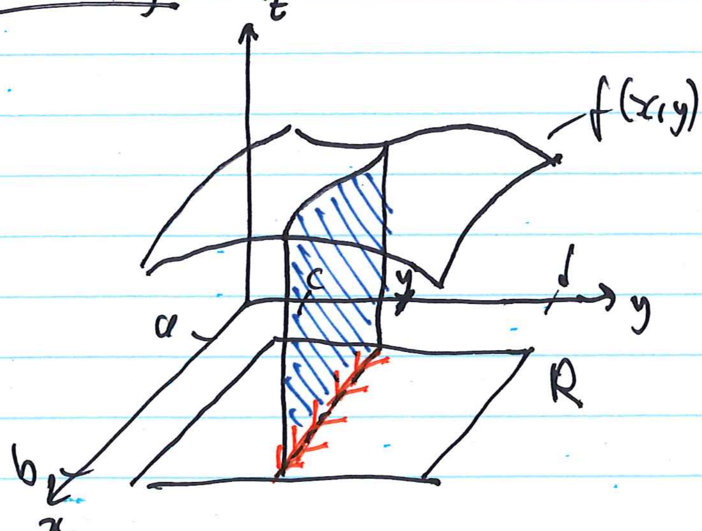
$$\int_0^1\left(\int_1^2 (1-x)y\, dy\right)dx = \int_0^1\left((1-x)\int_1^2 y\, dy\right)dx$$
$$= \int_0^1 (1-x)\, \frac{y^2}{2}\Big|_{y=1}^{y=2} dx$$
$$= \int_0^1 (1-x)\cdot\frac{3}{2}\, dx$$
$$= \frac{3}{2}\left(-\frac{(1-x)^2}{2}\Big|_0^1\right)$$
$$= 3/4$$



Sol'n 2

$$\int_1^2\int_0^1 (1-x)y\, dx\, dy = \int_1^2 y\left(-\frac{(1-x)^2}{2}\Big|_{x=0}^{x=1}\right)dy$$
$$= \frac{1}{2}\int_1^2 y\, dy = 1/2\ \frac{y^2}{2}\Big|_1^2$$
$$= 3/4$$

Geometry



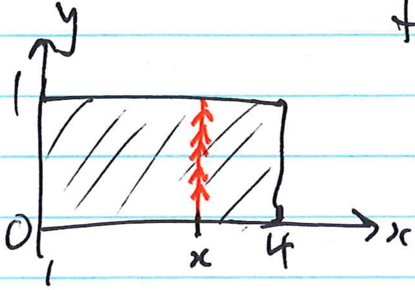
$$\int_c^d \underbrace{\left[\int_a^b f(x,y)\, dx\right]}_{A(y)} dy$$

$A(y)$: area under the trace curve for fixed y.

$$\text{Volume} = \int_c^d A(y)\, dy$$

~~Goo~~ Google: J Lo Horse Scene

<u>Example</u> find the average value of
$f(x,y) = e^y \sqrt{x+e^y}$ over the rectangle
$R = [0,4] \times [0,1]$



$$f_{avg} = \underbrace{\frac{1}{4}}_{\text{area}(R)} \iint_R f(x,y)\, dA = \frac{1}{4} \int_0^4 \left(\int_0^1 e^y \sqrt{x+e^y}\, dy \right) dx$$

<u>Subs</u> $u = x + e^y \qquad u(y=0) = x+1$
$du = e^y dy \qquad u(y=1) = x+e$

$$\int_0^1 e^y \sqrt{x+e^y}\, dy = \int_{x+1}^{x+e} \sqrt{u}\, du = \frac{2}{3} u^{3/2} \Big|_{x+1}^{x+e}$$
$$= \frac{2}{3}\left[(x+e)^{3/2} - (x+1)^{3/2} \right]$$

Recall $f_{avg} = \frac{1}{4} \int_0^4 \frac{2}{3}\left[(x+e)^{3/2} - (x+1)^{3/2} \right] dx$

$$= \frac{1}{6} \left[\frac{2}{5}(x+e)^{5/2} - \frac{2}{5}(x+1)^{5/2} \right]_{x=0}^{x=4} = \ldots$$

≈ 3.3

Example: evaluate $I = \iint_R x^y \, dx \, dy$ $R = [0,1] \times [0,1]$

$$I = \int_0^1 \left(\int_0^1 x^y \, dx \right) dy = \int_0^1 \frac{1}{y+1} x^{y+1} \Big|_{x=0}^{x=1} dy$$

Carefully would have limit

$$= \int_0^1 \frac{1}{y+1} \, dy$$

$$= \ln(y+1) \Big|_0^1 = \ln 2$$

$$I = \int_0^1 \int_0^1 x^y \, dy \, dx = \int_0^1 \left(\int_0^1 e^{y \ln x} \, dy \right) dx$$

$$= \int_0^1 \frac{1}{\ln x} e^{y \ln x} \Big|_{y=0}^{y=1} dx$$

$$= \int_0^1 \frac{1}{\ln x} \left(e^{\ln x} - 1 \right) dx$$

$$= \int_0^1 \frac{x-1}{\ln x} \, dx = \text{yikes!}$$

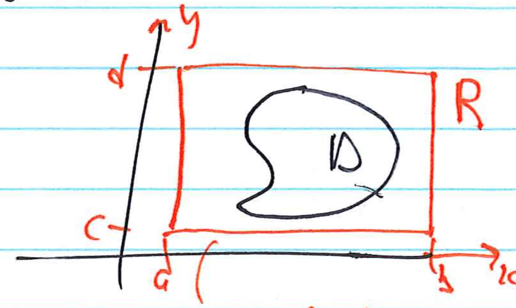
Sometimes one of $dx\,dy$ or $dy\,dx$ $\iint_R$ is easier!

Double integrals ~~of~~ over general domains

§13.1

Let D be a closed, bounded region of the x-y plane

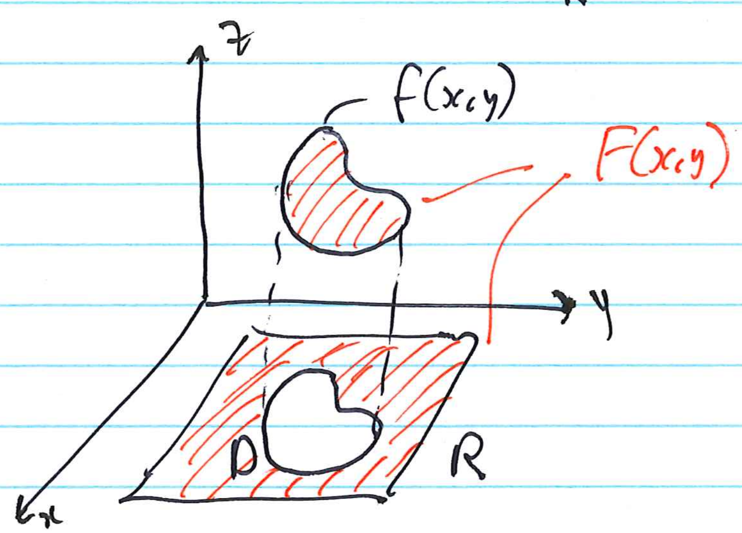
We want $\displaystyle\iint_D f(x,y)\,dA$



R rectangle contains D.

Def'n: $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\notin D \end{cases}$

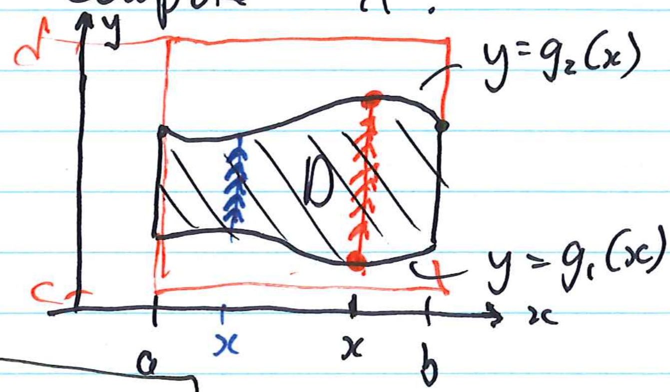
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$



Note: F discontinuous but $\displaystyle\iint_R F\,dA$ exists provided boundary of D not too "wild".

Geometric interpretation of $\displaystyle\iint_D f\,dA$ is the volume over D and under $f(x,y)$

Ok, but how to compute it?

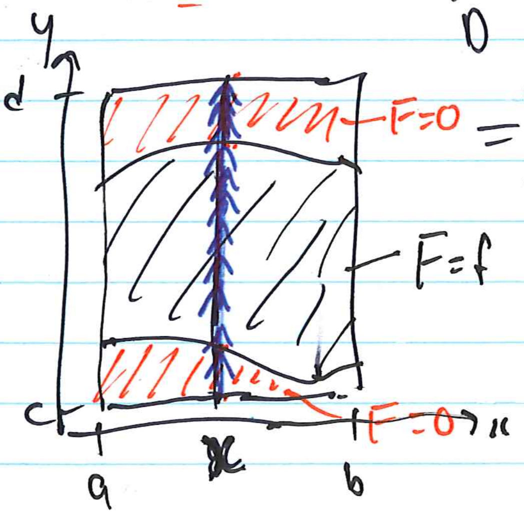


Type I region

$$D = \{(x,y) : \; a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

Derivation: $\displaystyle \iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$



$$= \int_a^b \left(\int_c^d F(x,y)\,\cancel{dA}\,dy\,dx \right) = \int_a^b \left[\int_c^{g_1(x)} \overset{0}{F}\,dy + \int_{g_1(x)}^{g_2(x)} \overset{f}{F}\,dy + \int_{g_2(x)}^{d} \overset{0}{F}\,dy \right] dx$$