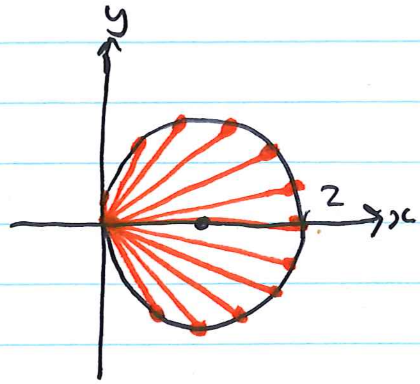
Last day: polar

Ex (last day): plot the curve $r(\theta) = 2\cos\theta$ in the xy-plane.

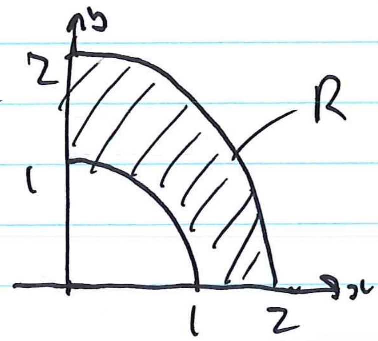


Algebraic answer:

$$r = 2\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$r^2 = 2x$$
$$x^2 + y^2 = 2x$$
$$x^2 - 2x + 1 + y^2 = 1$$
$$(x-1)^2 + y^2 = 1$$

From computer demo, and from $\cos\theta$ values note $\theta \in [-\frac{\pi}{2}, \pi/2]$

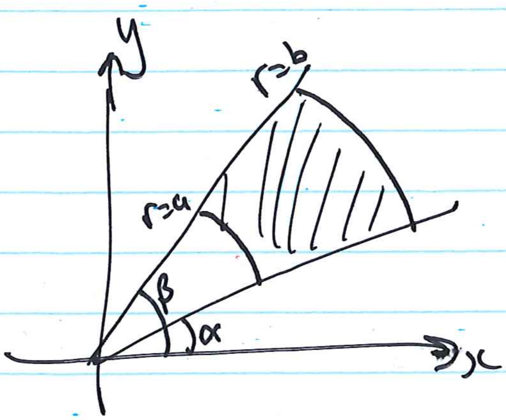
Ex



Express region R of x-y plane in polar coordinates.

$$R = \{(r,\theta) : 1 \le r \le 2, \; 0 \le \theta \le \pi/2\}$$

Defn: polar rectangle

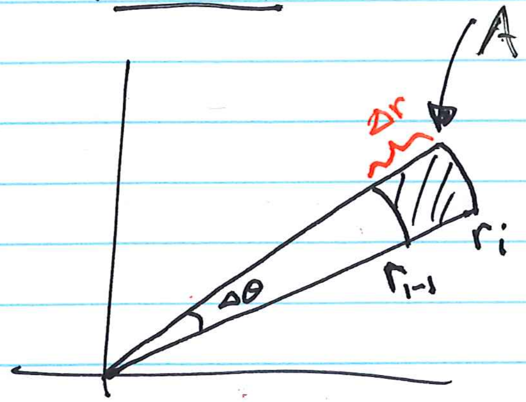


$$R = \{(r,\theta) : a \le r \le b, \; \alpha \le \theta \le \beta\}$$

Goal: $\iint_R f(x,y)\, dA$

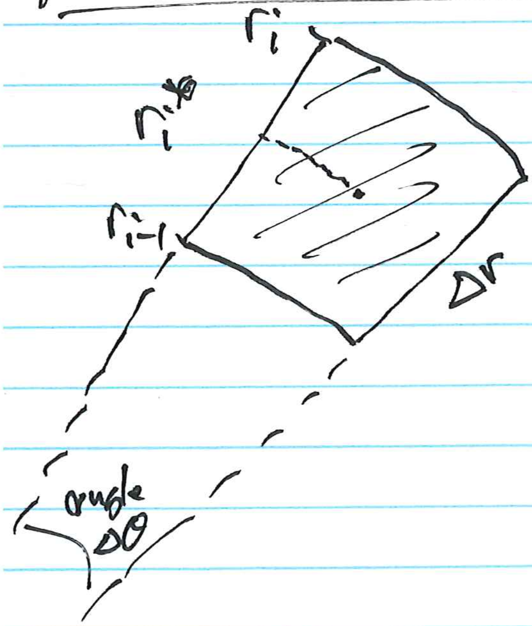
where R is a polar rectangle.

Lemma Area of a small polar rectangle.



$$A = \frac{1}{2}\Delta\theta \underbrace{\left(\frac{r_i + r_{i-1}}{2}\right)}_{r_i^*} \underbrace{(r_i - r_{i-1})}_{\Delta r}$$

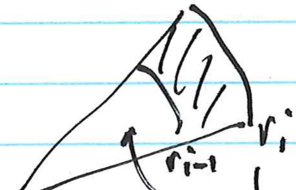
$$\boxed{A = r_i^* \Delta\theta \Delta r}$$



Note: area depends on r

Area of:

$$\pi r_i^2 \left(\frac{\Delta\theta}{2\pi}\right)$$



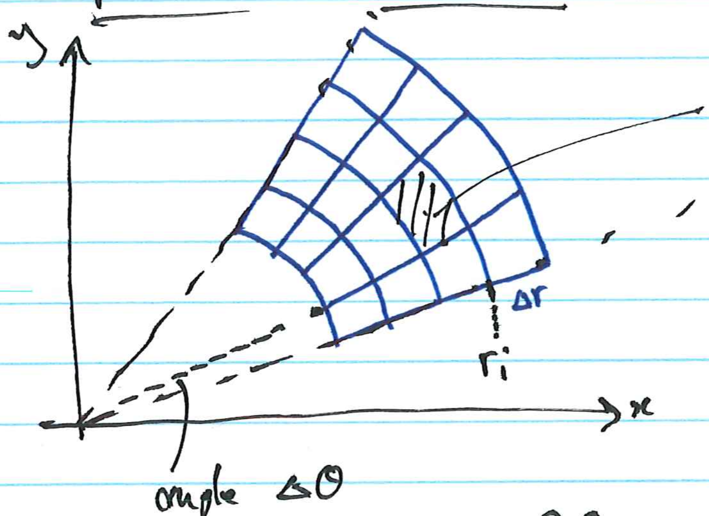
eat this bit

Area remaining is

$$\frac{\Delta\theta}{2\pi}\pi r_i^2 - \frac{\Delta\theta}{2\pi}\pi r_{i-1}^2$$

$$= \frac{1}{2}\Delta\theta\left(r_i^2 - r_{i-1}^2\right)$$

Polar Riemann over region $R = \{ a \le r \le b, \ \alpha \le \theta \le \beta \}$



area $\Delta A_i = r_i^* \Delta\theta \Delta r$

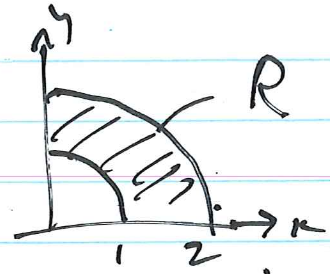
(by lemma)

We want $\displaystyle\iint_R f(x,y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i \cos\theta_j, r_i \sin\theta_j) \cdot \Delta A_i$

$$= \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i \cos\theta_j, r_i \sin\theta_j)\, r_i^* \Delta\theta \Delta r$$

$$\boxed{\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, \underbrace{r\, dr\, d\theta}_{dA}}$$

Ex $\quad I = \iint_R x^2 \, dA$



$$I = \int_0^{\pi/2}\int_1^2 \underbrace{(r\cos\theta)^2 \, r}_{\text{integrand}} \, dr \, d\theta$$

$$R = \left\{ \overset{a}{1} \le r \le \overset{b}{2}, \quad \underset{\alpha}{0} \le \theta \le \underset{\beta}{\pi/2} \right\}$$

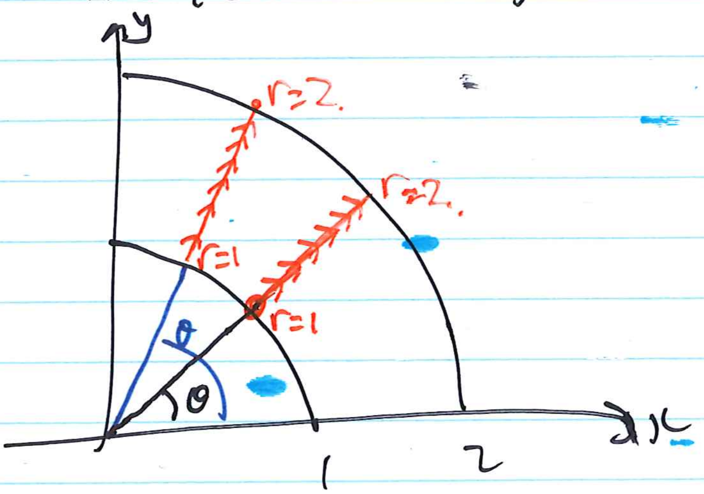
$$= \int_0^{\pi/2} \cos^2\theta \underbrace{\left[\int_1^2 r^3 \, dr \right]}_{\frac{1}{4} r^4 \Big|_1^2 = \frac{15}{4}} d\theta$$

double angle law

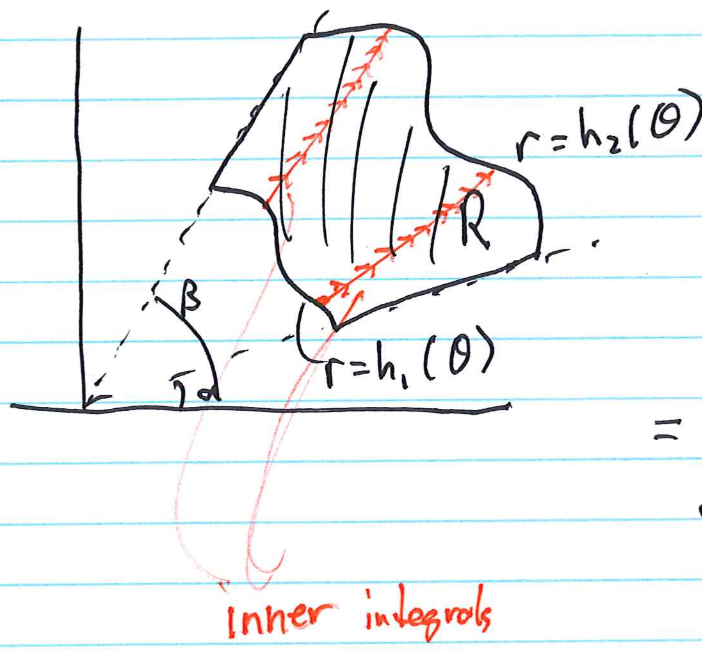
$$= \int_0^{\pi/2} \frac{15}{4} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{15}{4} \left(\frac{\cos 2\theta + 1}{2} \right) d\theta$$

$$= \frac{15}{16}\pi$$

Inner/Outer integral in this problem?



Irregular regions

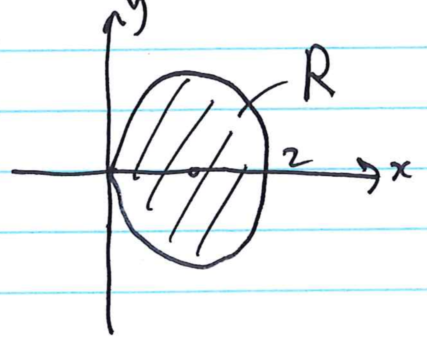


$$R = \{ \alpha \le \theta \le \beta, \text{ and } h_1(\theta) \le r \le h_2(\theta) \}$$

$$\iint_R f(x,y)\,dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex Find the volume below paraboloid $z = x^2+y^2$ above the x-y plane and inside the cylinder $(x-1)^2+y^2 = 1$



$$R = \{ (x,y) : (x-1)^2+y^2 \le 1 \}$$

$$\iint_R \underbrace{x^2+y^2}_{r^2} \; \underbrace{dA}_{r\,dr\,d\theta}$$

Recall (from earlier)
$R = \{ (r,\theta) :$
$-\pi/2 \le \theta \le \pi/2$
$0 \le r \le 2\cos\theta \}$