

**Midterm 1      Duration: 50 minutes**

*This test has 4 questions on 6 pages, each worth 10 points, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

6 marks

1. (a) Find the equation of the plane containing the point  $(2, 0, -1)$  and containing the line  $x = 1 - t$ ,  $y = 2t$ ,  $z = 2 + 2t$ .

$$\text{Answer: } 6x + y + 2z = 10$$

**Solution:** Let  $P$  be the point  $(2, 0, -1)$ . We set  $t = 0$  and  $t = 1$  to see that two points on the line are  $Q = (1, 0, 2)$  and  $R = (0, 2, 4)$ . Two vectors in the plane are  $\overrightarrow{PQ} = \langle -1, 0, 3 \rangle$  and  $\overrightarrow{PR} = \langle -2, 2, 5 \rangle$ . Their cross product is  $\langle -1, 0, 3 \rangle \times \langle -2, 2, 5 \rangle = \langle -6, -1, -2 \rangle$ , and this vector is normal to the plane. The vector  $\langle 6, 1, 2 \rangle$  is also normal. The equation of the plane therefore has the form  $6x + y + 2z = d$ . We substitute  $(x, y, z) = (2, 0, -1)$  to find that  $d = 10$ , so the plane is  $6x + y + 2z = 10$ .

2 marks

- (b) Find the parametric equations of the line which is normal to the plane  $x + 2z = 1$  and which contains the point  $(0, -1, 0)$ .

$$\text{Answer: } x = t, y = -1, z = 2t$$

**Solution:** The plane has normal  $\langle 1, 0, 2 \rangle$  and this defines the direction of the line, whose parametric equations are therefore  $x = t$ ,  $y = -1$ ,  $z = 2t$ .

2 marks

- (c) Find the acute angle between the planes  $x + y = -5$  and  $y - z = 12$ .

$$\text{Answer: } \frac{\pi}{3}, \text{ or } 60^\circ$$

**Solution:** Normal vectors to the planes are  $\vec{n}_1 = \langle 1, 1, 0 \rangle$  and  $\vec{n}_2 = \langle 0, 1, -1 \rangle$ . The angle between the planes is the angle  $\theta$  between the normals, which obeys

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Therefore  $\theta = \frac{\pi}{3}$ , or  $60^\circ$ .

2 marks

2. (a) Let  $f(x, y) = e^{xy} + x^2y + \cos(x)$ . Compute the partial derivatives  $f_x$  and  $f_y$ .

$$\begin{aligned} \text{Answer: } f_x &= ye^{xy} + 2xy - \sin x, \\ f_y &= xe^{xy} + x^2. \end{aligned}$$

**Solution:** Partial differentiation simply gives  $f_x = ye^{xy} + 2xy - \sin x$ ,  $f_y = xe^{xy} + x^2$ .

4 marks

- (b) Find all values of the constant  $s$  such that  $f(x, y) = e^{sx} \cos(3y)$  satisfies the Laplace equation  $f_{xx} + f_{yy} = 0$ .

$$\text{Answer: } s = \pm 3$$

**Solution:** We compute the partial derivatives:

$$\begin{aligned} f_x &= se^{sx} \cos(3y), & f_{xx} &= s^2 e^{sx} \cos(3y), \\ f_y &= -3e^{sx} \sin(3y), & f_{yy} &= -9e^{sx} \cos(3y). \end{aligned}$$

Therefore,

$$f_{xx} + f_{yy} = (s^2 - 9)e^{sx} \cos(3y),$$

and this equals zero when  $s^2 = 9$ , i.e.,  $s = \pm 3$ .

4 marks

- (c) The equation  $z^3 - z + 2xy - y^2 = 0$  determines a function  $z = f(x, y)$  implicitly near  $(x, y) = (2, 4)$  with  $f(2, 4) = 1$ . Find the derivative  $\frac{\partial z}{\partial y}$  at the point  $(x, y) = (2, 4)$ .

$$\text{Answer: } 2$$

**Solution:** We differentiate implicitly to obtain

$$3z^2 \frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} + 2x - 2y = 0,$$

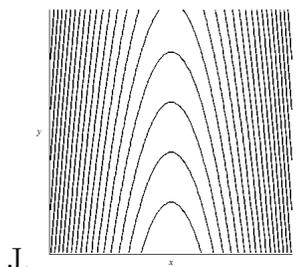
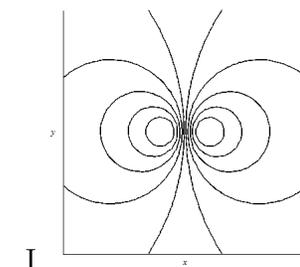
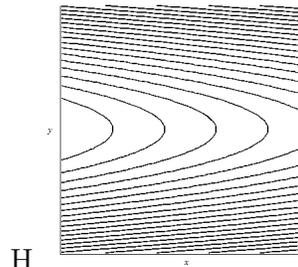
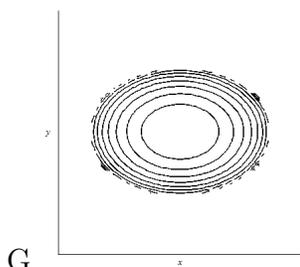
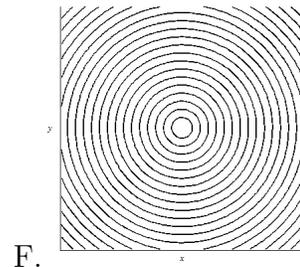
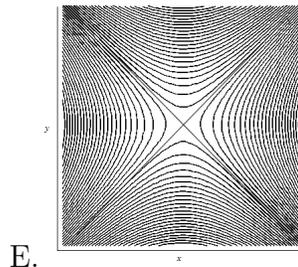
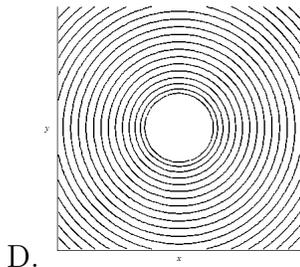
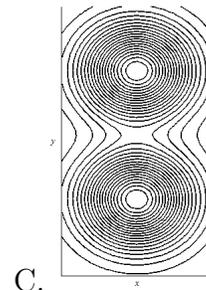
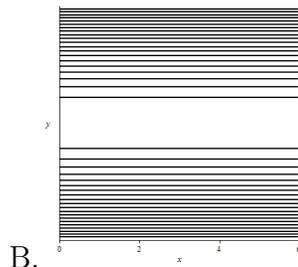
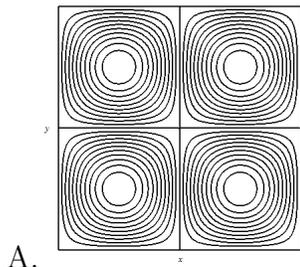
which gives

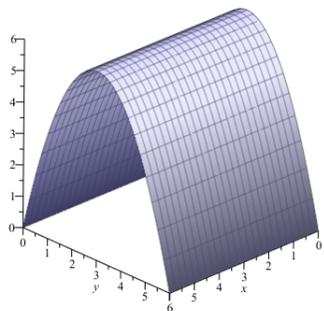
$$\frac{\partial z}{\partial y} = -\frac{2x - 2y}{3z^2 - 1}.$$

At  $(x, y) = (2, 4)$ , where  $z = 1$ , the right-hand side is 2.

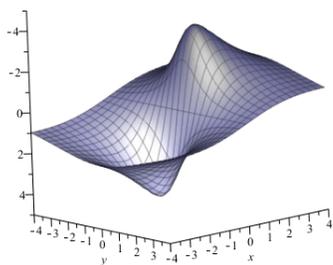
10 marks

3. Consider the following 10 contour plots, 5 graphs, and 5 functions. Each contour plot is the contour plot of one of the 5 graphs, or of one of the 5 functions. Match each contour plot with the corresponding graph or function. In the 10 contour plots, the  $x$  axis is horizontal and the  $y$  axis is vertical. In the 5 graphs, the  $z$  axis is vertical, and the  $y$  and  $x$  axes are the other two with the  $x$  axis rightmost.

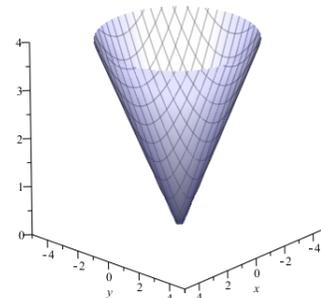




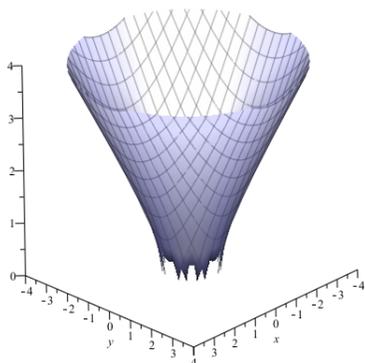
1.



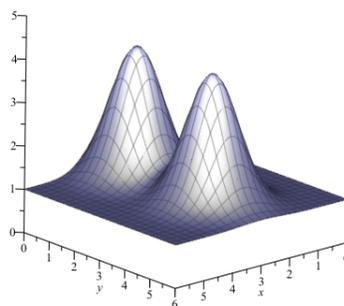
2.



3.



4.



5.

6.  $f(x, y) = \sqrt{1 - 3x^2 - 6y^2}$ .

7.  $f(x, y) = 3y + 2x^2$ .

8.  $f(x, y) = 1 - x^2 + y^2$ .

9.  $f(x, y) = 1 + \sin(x) \sin(y)$ .

10.  $f(x, y) = 3x + 2y^2$ .

Write your answer in order as  $Aa, Bb, Cc, \dots, Jj$  where  $a$  is the number of the function corresponding to contour plot  $A$ ,  $b$  to  $B$ , and so on.

Answer: A9, B1, C5, D4, E8, F3, G6, H10, I2, J7

5 marks

4. (a) Find the equation of the plane tangent to  $z = \sqrt{36 - 4x^2 - 9y^2}$  at the point  $(x, y, z) = (2, 1, \sqrt{11})$ .

$$\text{Answer: } z = -\frac{1}{\sqrt{11}}(8x + 9y - 36)$$

**Solution:** The tangent plane has equation  $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ . Computation gives  $f_x = -4x/z$ ,  $f_y = -9y/z$ , and substitution then gives

$$z = \sqrt{11} - \frac{8}{\sqrt{11}}(x - 2) - \frac{9}{\sqrt{11}}(y - 1) = -\frac{1}{\sqrt{11}}(8x + 9y - 36).$$

5 marks

- (b) Three positive numbers, each less than 10, are multiplied together. The values of the three numbers are known with respective errors at most 0.1, 0.05 and 0.04. *Using differentials*, estimate the maximum possible error in the computed product.

$$\text{Answer: } 19$$

**Solution:** Let  $x, y, z$  be the three numbers, and let  $P = xyz$  be their product. Then

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz = yzdx + xzdy + xydz.$$

This is maximal when  $x = y = z = 10$ , with  $dx = 0.1$ ,  $dy = 0.05$ ,  $dz = 0.04$ , in which case it equals  $10^2(0.1 + 0.05 + 0.04) = 19$ .