THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253
Midterm 2
14 November 2012

TIME: 50 MINUTES

FIRST NAME: Solution LAST NAME : ____________

STUDENT #: ____________

This Examination paper consists of 7 pages (including this one). Make sure you have all 7.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.
PLEASE CIRCLE YOUR INSTRUCTOR'S NAME BELOW

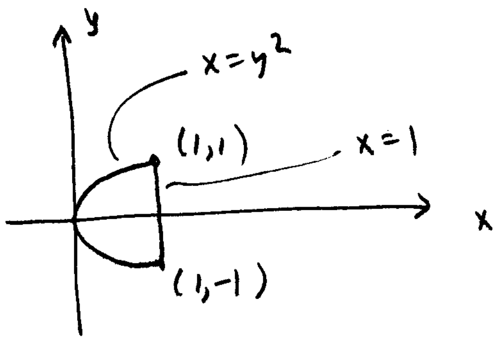
MARKING:

Q1	/10
Q2	/12
Q3	/12
Q4	/16
TOTAL	/50

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Ian Hewitt, Yariv Dror-Mizrahi, Ed Richmond

Q1 [10 marks]

Find the volume of the region in 3-space which is below the surface $z = 1 + 3x^2y^2$ and lies above the region in the xy-plane enclosed by the curves $x = y^2$ and $x = 1$.



$$\text{Volume} = \int_{y=-1}^{1} \int_{x=y^2}^{1} \left(1 + 3x^2y^2\right) dx\, dy$$

$$= \int_{y=-1}^{1} \left[x + x^3y^2\right]_{y^2}^{1} dy$$

$$= \int_{y=-1}^{1} \left(1 + \cancel{y^2} - \cancel{y^2} - y^8\right) dy = \left[y - \tfrac{1}{9}y^9\right]_{-1}^{1}$$

$$= \left(1 - \tfrac{1}{9}\right) - \left(-1 + \tfrac{1}{9}\right) = 2 - \tfrac{2}{9} = \tfrac{16}{9}$$

Q2 [12 marks]

Suppose $T(x,y,z) = xy^2 - x + x^2z + yz^2$ gives the temperature at the point (x,y,z) in space.

(a) Find an equation of the plane tangent at $(1,2,1)$ to the level surface of T passing through that point.[4pts]

$$\vec{\nabla} T = \langle y^2 - 1 + 2xz,\ 2xy + z^2,\ x^2 + 2yz \rangle$$

$$\vec{\nabla} T(1,2,1) = \langle 4 - 1 + 2,\ 4 + 1,\ 1 + 4 \rangle = \langle 5,5,5 \rangle$$

eqn of tangent plane: $\underline{5(x-1) + 5(y-2) + 5(z-1) = 0}$

or $\underline{(x-1) + (y-2) + (z-1) = 0}$

or $\underline{x + y + z = 4}$

(b) At time $t = 0$, a fly passes through $(1,2,1)$ moving toward the point $(4,2,5)$ at speed of 1 unit/sec. Calculate $\dfrac{dT}{dt}$ at $t = 0$ for the fly.[4pts]

direction of fly: $\langle 4,2,5 \rangle - \langle 1,2,1 \rangle = \langle 3,0,4 \rangle$

unit vector $\vec{u} = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$

$$\left.\frac{dT}{dt}\right|_{t=0} = \vec{\nabla} T \cdot \vec{u} = \langle 5,5,5 \rangle \cdot \langle \tfrac{3}{5}, 0, \tfrac{4}{5} \rangle = 7$$

(c) A worm crawling on the plane $2x - y + 2z = 2$ passes through the point $(1, 2, 1)$. The worm wishes to keep his temperature constant while increasing z. In which direction should the worm move? Express your answer as a unit vector. [4pts]

direction of worm is perpendicular to $\langle 2,-1,2\rangle$ since worm crawls on the plane $2x-y+2z=2$

direction of worm is perpendicular to $\langle 5,5,5\rangle$ since worm wants his temperature constant.

Thus we seek a unit vector, with positive $\vec{k}$ component, which is perpendicular to $\langle 2,-1,2\rangle$ & $\langle 5,5,5\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 5 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 5\langle -3, 0, 3\rangle$$

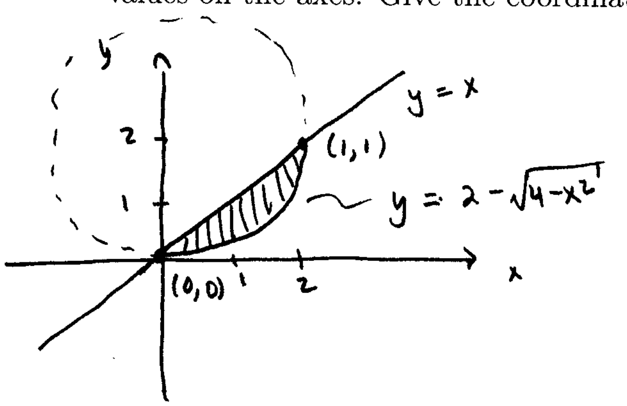
make it a unit vector $\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

Q3 [12 marks]

Consider the following iterated integral

$$\int_0^2 \int_{2-\sqrt{4-x^2}}^{x} f(x,y)\,dydx.$$

(a) Sketch the region of integration. Be sure to label your axes and clearly mark x and y values on the axes. Give the coordinates of any intersection points. [4pt]



$y = 2 - \sqrt{4-x^2}$

$\Rightarrow (y-2)^2 = 4 - x^2$

$\Rightarrow (y-2)^2 + x^2 = 2^2$ — circle of radius 2 centered at $(0,2)$

$\Rightarrow x^2 = 4 - (y-2)^2$

$x = \sqrt{4-(y-2)^2}$

(b) Change the order of integration to $dx\,dy$. [4pt]

$$\int_{y=0}^{2} \int_{x=y}^{\sqrt{4-(y-2)^2}} f(x,y)\,dx\,dy$$

(c) Convert the integral to polar coordinates. [4pt]

$$\int_{\theta=0}^{\pi/4} \int_{r=0}^{4\sin\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$y^2 - 4y + \cancel{4} + x^2 = \cancel{4}$

$y^2 + x^2 = 4y$

$r^2 = 4r\sin\theta$

$r = 4\sin\theta$

Q4 [16 marks]

Consider the function $f(x,y) = (4y+7)e^{-x^2-y^2}$ on the domain $x^2+y^2 \leq 1$.

(*a*) Find all critical points of f which are inside the domain [4 pts]

$f_x = -2x(4y+7)e^{-x^2-y^2}$ $\qquad f_y = 4e^{-x^2-y^2} - 2y(4y+7)e^{-x^2-y^2}$

$f_x = 0 \Rightarrow -2x(4y+7)=0$ $\qquad f_y = 0 \Rightarrow 4 - 2y(4y+7) = 0$

$\Downarrow$

$x=0$ or $4y+7=0$ this doesn't work since plugging in here gives $4=0$

so $x=0$ and $4-2y(4y+7)=0 \Rightarrow 4-8y^2-14y=0 \Rightarrow 8y^2+14y-4=0$

$\Rightarrow (8y-2)(y+2)=0 \Rightarrow y=-2$ or $\frac{1}{4}$ critical points are $(0,-2)$ and $(0,\frac{1}{4})$ only $\underline{\underline{(0,\frac{1}{4})}}$ is in the domain

(*b*) Classify each of the critical points on the inside of the domain as a "local maximum", "local minimum", "saddle points", or "discriminant is zero". [4 pts]

$f_{xx} = -2(4y+7)e^{-x^2-y^2} + 4x^2(4y+7)e^{-x^2-y^2}$

~~$f_{yy} = -8ye^{-x^2-y^2} \ldots$~~ $\qquad f_y = (4-2y(4y+7))e^{-x^2-y^2}$

$= (4-8y^2-14y)e^{-x^2-y^2}$

$f_{yy} = (-16y-14)e^{-x^2-y^2} - 2y(4-8y^2-14y)e^{-x^2-y^2}$

$f_{xy} = -2x(4-8y^2-14y)e^{-x^2-y^2}$

$f_{xx}(0,\frac{1}{4}) = -2(1+7)e^{-\frac{1}{16}} = -16e^{-\frac{1}{16}}$

$f_{yy}(0,\frac{1}{4}) = (-4-14)e^{-1/16} - \frac{2}{4}(4-\frac{1}{2}-\frac{7}{2})e^{-1/16} = -18e^{-1/16}$

0

$f_{xy}(0,\frac{1}{4}) = 0$ so $D = f_{xx}f_{yy} - f_{xy}^2 = (-16)e^{-\frac{1}{16}}(-18)e^{-1/16} > 0$

and $f_{xx} < 0 \Rightarrow (0,\frac{1}{4})$ is a $\underline{\text{local maximum}}$

(c) Use the method of Lagrange multipliers to find the maximum and minimum values of f on the boundary of the domain. [6pt]

① $g(x,y) = x^2+y^2 = 1 \quad f_x = \lambda g_x \quad f_y = \lambda g_y$

② $-2x(4y+7)e^{-x^2-y^2} = \lambda 2x \Rightarrow x=0$ or $\lambda = -(4y+7)e^{-x^2-y^2}$

③ $(4-8y^2-14y)e^{-x^2-y^2} = \lambda 2y$

$\Downarrow$ sub into ③

$(4-8y^2-14y)e^{-x^2-y^2} = -2y(4y+7)e^{-x^2-y^2}$

$\Downarrow$

$4-8y^2-14y = -8y^2-14y \Rightarrow 4=0$

no solution

$x=0$ sub into ① $\Rightarrow y = \pm 1$

so $(x,y) = (0,1), (0,-1)$ are solutions

so $(0,1)$ & $(0,-1)$ are the only possibilities

$\underbrace{f(0,1) = 11e^{-1}}_{\text{max on boundary}} \quad \underbrace{f(0,-1) = 3e^{-1}}_{\text{min on boundary}}$

(d) Find the absolute maximum and minimum values of f on its whole domain. You may use the fact that $e^{15/16} > 11/8$. [2 pts]

Abs. max/min must occur at a crit pt on the interior or a max/min on the boundary. So only possibilities are $(0,1)$, $(0,-1)$, $(0,\frac{1}{4})$

$f(0,1) = 11e^{-1}$

$f(0,-1) = 3e^{-1}$

$f(0,\frac{1}{4}) = 8e^{-1/16}$

since $e^{15/16} > 11/8$

$\Rightarrow 8e^{1-1/16} > 11$

$\Rightarrow 8e^{-1/16} > 11e^{-1}$

so $3e^{-1}$ is absolute min and $8e^{-1/16}$ is absolute max.