Math 253 2017: Homework 3

1. **Laplace's equation.** Indicate with a "yes" or "no" whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

   ☐ (a) $u = x^2 + y^2$

   ☐ (b) $u = x^2 - y^2$

   ☐ (c) $u = x^3 + 3xy^2$

   ☐ (d) $u = \ln \sqrt{x^2 + y^2}$

   ☐ (e) $u = \sin x \cosh y + \cos x \sinh y$

   ☐ (f) $u = e^{-x} \cos y - e^{-y} \cos x$

2. **Ch-ch-ch-Chain Rule.** Consider $f(x, y)$, where x and y are functions of r and θ: $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. The chain rule expresses $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of f_x and f_y; it can be written in matrix form:

$$\begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

(a) Give the 2×2 matrix J explicitly. Each entry may depend on r and/or θ.

(b) What is $\det(J)$? What is J^{-1}? (The matrix inverse, entries should depend on r and/or θ).

(c) Now suppose $g(r,\theta)$, with $r(x,y)$ and $\theta(x,y)$. These satisfy

$$(r(x,y))^2 = x^2 + y^2, \qquad \tan(\theta(x,y)) = y/x.$$

Work out the four partials $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$, and $\frac{\partial \theta}{\partial y}$, expressing your answers in (only) r and θ. Hint: implicit differentiation is your friend. Hint 2: keep track of who is an independent variable and who is a dependent variable. Hint 3: if the $\tan\theta$ formula is difficult, it might be easier to use $r\cos\theta = x$.

(d) Use the chain rule to express $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in terms of the $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$. Write this chain rule for g in a matrix form, using a 2×2 matrix B. Express entries of B as functions of r and/or θ.

(e) The Liebniz notation $\frac{\partial x}{\partial r}$ makes it tempting to think $\frac{\partial x}{\partial r} = \frac{1}{\frac{\partial r}{\partial x}}$. Is this so?

(f) Compute $\det(B)$.

(g) How does this relate to $\det J$? How does B relate to J^{-1}?

3. **A mixed-partial roof is better than a mixed partial-roof.** UBC's First Nations Longhouse has an interesting roof. Let's reverse engineer it. Suppose the Longhouse occupies a rectangle in the xy-plane, $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$. The roof is supported by long straight beams across the narrower width of the building; we can represent a beam using an equation $z = mx + b$.

   (a) Let's suppose the beam at $y = -2$ has equation $z = -\frac{1}{2}x + 1$. Make a sketch in the xz-plane showing this beam and the wall of the Longhouse. At the opposite end of the Longhouse at $y = 2$, the beam is $z = \frac{1}{2}x + 1$. Make another sketch showing this wall and beam in the xz-plane. Label your sketches with $y = -2$ and $y = 2$.

   (b) What should we do with the beams in between? Each one has an equation like $z = m_i x + 2$, but maybe that's a tad too discrete for this course. How about $m(y)$: different m for different y values. What is the simplest function $m(y)$ with $m(-2) = -\frac{1}{2}$ and $m(2) = \frac{1}{2}$?

   (c) Hence, give an expression for the roof surface $z = f(x, y)$.

   (d) Compute the partial derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy} (and f_{yx} if you don't yet trust Signore Professore Fubini).

   Now would be an appropriate time to contemplate the meaning of $\frac{\partial}{\partial y} f_x$, also known as f_{xy}. Share your "aha!" moment with a drink and a friend.

(e) Make an attractive computer plot—or an *extremely attractive* sketch—of the surface $f(x, y)$.

(f) Consider the surface $z = g(x, y)$ with $g(x, y) = cxy + D(x) + E(y) + F$ where D, E are *any* differentiable functions of one variable and F is a constant. Show that expressions of this form satisfy $g_{xy} = c$.

(g) Postulate (guess) a general solution to the system of partial differential equations given by

$$g_{xy} = c, \qquad g_{xx} = 0, \qquad g_{yy} = 0.$$

These sorts of calculations are related to *minimal energy surfaces*, such as soap bubbles. And, it would seem, to constrained creativity in architecture.