Math 405: Topic 5: Intro to Finite Differences

References: LeVeque 2007, Finite Difference Methods for ODE and PDE. Particularly Chapters 1, 2, and 9.

Others: Morton and Mayers, Suli and Mayers. Many specialized textbooks.

Finite Differences

Very general technique for differential equations, based on approximating functions on a grid and replacing *derivatives* with *differences*.

Example: definition of derivative. Drop limit, take h finite.

Somehow, everything we want to talk about is already present in this example:

- continuous –> discrete
- differential –> algebraic
- accuracy (Taylor series analysis, effect of terms dropped?)
- division by small h
- computational issues: want more accuracy? use small h, more points, more work.

Approximating functions on a grid

Notation: replace function $u(x)$ with some discrete U_j, $j = 0, \dots m$. $U_j \approx u(x_j)$ on a discrete set of points x_j.

I reserve the right to also use both $u_j = U_j$ and to change index variables at will ;-)

For multidimensional (e.g., time dep.) problem: $U_j^n \approx u(x_j, t_n)$.

A side note on interpolation

How to derive finite difference rules like the example? One approach: fit a polynomial $ax^2 + bx + c$ through the data, the differentiate that polynomial.

Fitting such a polynomial is *interpolation*. The monomial basis is often a poor choice [sketch motivation], instead look at the "cardinal polynomial":

$$L_{m,j}(x) = L_j(x) := \frac{(x - x_0) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_m)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_m)},$$

These are like the "$[0; 1; 0; 0]$" vectors: zero at all grid points except x_j.

Polynomial interpolant of degree m:

$$p(x) := \sum_{j=0}^{m} u_j L_j(x)$$

Very nice mathematical theory here—we've seen a bit of it—$!\exists$, etc. Also useful in practice.

Analysis of the first-derivative

When doing quadrature, we could do (rough) error analysis based on the error formula for interpolant. Can we do so here?

[Example: no, at least not in the most obvious way. . .]

Second derivative

The derivative of the derivative: use backward difference of the forward difference.

$$u_{xx}(x_j) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

or we can say more with a few more symbols:

$$u_{xx}(x_j) = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + O(h^2)$$

This will be our “work-horse” method/example.

Note: **second-order accurate**: replace h with $h/2$, error should divide by four (for smooth enough function).

As a matrix

Suppose x on interval $[a, b]$ and end values are $u(a) = u_0 = \alpha$ and $u(b) = u_{m+1} = \beta$. Applying the “1 -2 1” rule at each point gives:

```
u_{xx} \approx

           |-2    1                 |   | u_1 |      | alpha/h^2 |
           | 1   -2    1            |   | u_2 |      |     0     |
           |    .     .     .       |   |  .  |      |     .     |
     1/h^2 |       .     .     .    |   |  .  |  +   |     .     |
           |          .     .     . |   |  .  |      |     .     |
           |            1    -2    1 |   |     |      |     0     |
           |                  1   -2 |   | u_m |      | beta/h^2  |
```

(other ways to view this, e.g., as a rectangular matrix).

Evaluating a differential operator on a scalar function is approximated by a matrix multiplying a vector.

Calculus becomes linear algebra

Heat/Diffusion Equation

(All with initial conditions and boundary conditions as appropriate.)

Our primary example problem: $u_t = u_{xx}$

Or with a forcing: $u_t = \kappa u_{xx} + f$

Or the steady problem: $u_{xx} = f$ (different f)

Example:

```
|-2    1                 |   | u_1 |      | f_1 - alpha/h^2 |
| 1   -2    1            |   | u_2 |      | f_2             |
|    .     .     .       |   |  .  |      |  .              |
```

```
1/h^2 |     .    .    .     |  |  .  |  =  |  .              |
      |       .    .    .   |  |  .  |     |  .              |
      |         1   -2    1 |  |     |     | f_{m-1}         |
      |              1   -2 |  | u_m |     | f_m -  beta/h^2 |
```

That is

$$AU = F,$$

for vectors U and F. U is the unknown pointwise values of u, F is the data, sampled pointwise.

Analysis of finite differences

Our "1 -2 1" rule was $O(h^2)$ accurate for evaluating the 2nd derivative. Will it still be so when U is unknown, as in $AU = F$? (Can think of this as an inverse problem.)

Need three concepts: *consistency*, *stability* and *convergence*.

Local truncation error

Substitute the true solution of the PDE into the discrete problem. Discrepancy is the *local truncation error (LTE)*.

symbol: τ

Express LTE in "big Oh" notation in h. (Exact solution unknown, assumed smooth).

Example: steady state.

True solution $u(x)$, sub in:

$$\tau_j = \frac{1}{h^2}\left(u(x_{j-1}) - 2u(x_j) + u(x_{j+1})\right) - f(x_j)$$

Now Taylor expand each, and use the PDE:

$$\tau_j = \frac{1}{12}h^2 u''''(x_j) + O(h^4)$$

Thus $\tau_j = O(h^2)$ (if smooth enough)

Consistency

LTE to zero as $h \to 0$.

Convergence

Define $\hat{U} = [u(x_0); u(x_1); ...u(x_m)]$, exact solution restricted to grid.

Global error: difference $E = U - \hat{U}$

Convergence: global error $E \to 0$ as $h \to 0$.

$LTE \to 0$ implies $E \to 0$? No, often not. The DE couples the LTE together. We need some notion that the error we make on each piece of LTE does not grow too much when "fed into" another part of the calculation.

Example Continue the steady state one. Note: $\tau = A\hat{U} - F$

So

$$AE = -\tau$$

In this case, relationship between global error and LTE involves properties of the matrix.

Stability

Steady state problem:

Defn 2.1 from LeVeque: Finite diff. applied to linear BVP gives $AU = F$, where each depends on h the mesh width. The method is *stable* if exists constant C (indep. of h)

$$||(A_h)^{-1}|| \leq C.$$

for sufficiently small h ($h < h_0$).

Time dependent: small errors (LTE) at one time should not increase too much as they propagate. (More later).

Fundamental Theorem of Finite Differences

(Maybe even "… of Numerical Analysis".)

Consistency plus Stability implies Convergence

Stability is the hard one, often this "theorem" needs many restrictions for particular classes of problems and methods. Also, often we play with what we mean by stability for problem/method.

Example: stability for BVP in the 2-norm

If matrix is symmetric (e.g., the one above) then 2-norm is spectral radius (maximum magnitude eigenvalue).

Inverse A^{-1} is has eigenvalues $\frac{1}{\lambda}$.

So we want smallest eigenvalue of A bounded away from zero as $h \to 0$. (note: matrix gets larger as h gets smaller).

E.g., [LeVeque pg21], can show for $h = 1/(m+1)$ with zero Dirichlet conditions,

$$\lambda_p = \frac{2}{h^2}(\cos p\pi h - 1), \quad p = 1, 2, \ldots, m.$$

In particular, $\lambda_1 = -\pi^2 + O(h^2)$, via—you guessed it—Taylor.