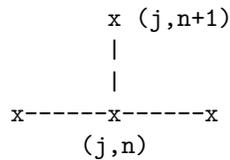


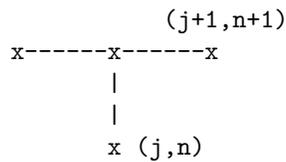
Math 405: Topic 7a: PDEs and the method-of-lines

Recall:

forward Euler: $v^{n+1} = v^n + kLv^n$



backward Euler: $v^{n+1} = v^n + kLv^{n+1}$



Time-step restrictions

Example: heat equation: disc./ in space with our favourite “1 -2 1” matrix.

We looked at eigenvalues before: they depend on h . Largest magnitude is $-4/h^2$. Need this inside the stability region.

Choose forward Euler: need $\lambda k > -2$.

Leads to restriction on k for stability in time:

$$k \leq \frac{1}{2}h^2$$

Stability in finite difference calculations

A fourth-order problem

$$u_t = -u_{xxxx}.$$

How to discretize? Think $(u_{xx})_{xx} \dots$, this leads, eventually, to

$$v_j^{n+1} = v_j^n - \frac{k}{h^4} (v_{j-2}^n - 4v_{j-1}^n + 6v_j^n - 4v_{j+1}^n + v_{j+2}^n)$$

[demo_07_biharmonic.m]

Note ridiculously small time steps required. Let's try to see why (a stability issue) and what we can do about it (implicit A-stable ODE methods).

von Neumann analysis

One approach is *von Neumann Analysis* of the finite difference formula, also known as *discrete Fourier analysis*, invented in the late 1940s.

Suppose we have periodic boundary conditions and that at step n we have a (complex) sine wave

$$v_j^n = \exp(i\xi x_j) = \exp(i\xi jh),$$

for some wave number ξ . Higher ξ is more oscillatory. We will analysis whether this wave grows in amplitude or decays (for each ξ). For stability, we want all waves to decay.

For the biharmonic diffusion equation, we substitute this wave into the finite difference scheme above, and factor out $\exp(i\xi h)$ to get

$$v_j^{n+1} = g(\xi)v_j^n,$$

with the *amplification factor*

$$g(\xi) = 1 - \frac{k}{h^4} (e^{-i2\xi h} - 4e^{-i\xi h} + 6 - 4e^{i\xi h} + e^{i2\xi h}).$$

This can be simplified to:

$$g(\xi) = 1 - \frac{16k}{h^4} (\sin(\xi h/2))^4.$$

As ξ ranges over various values \sin is bounded by 1 so we have

$$1 - 16k/h^4 \leq g(\xi) \leq 1.$$

A mode will grow if $|g(\xi)| > 1$. Thus for stability we want $|g(\xi)| \leq 1$ for all ξ , i.e.,

$$1 - 16k/h^4 \geq -1, \quad \text{or} \quad \boxed{k \leq h^4/8.}$$

For $h = 0.025$, as in the demo code, this gives

$$k \leq 4.883e - 08.$$

This matches our experiment convincingly, but confirms that this finite difference formula is not really practical.

Method-of-lines

As an alternative to von Neumann analysis, we follow the linear stability analysis for the ODE methods. The spatial discretization gives us (numerically anyway) the eigenvalues of the *semidiscrete* system. Need these eigenvalues to lie inside the absolute stability region of the ODE method.

Note: this involves the eigenvalues of the semidiscrete system, not the original right-hand-side of the PDE.

Demo: in Matlab, run `demo_07_biharmonic`, then use 'eigs' to compute 'largest magnitude' eigenvalues of the *discretized* biharmonic operator: need k times these less than 2 for forward Euler stability. Note this gives almost the same restriction as observed in practice (and calculated with von Neumann analysis)

