Math 405: Topic 7a: PDEs and the method-of-lines

Recall:

forward Euler: $v^{n+1} = v^n + kLv^n$

```
             x (j,n+1)
             |
             |
      x------x------x
           (j,n)
```

backward Euler: $v^{n+1} = v^n + kLv^{n+1}$

```
                 (j+1,n+1)
      x------x------x
             |
             |
             x (j,n)
```

Time-step restrictions

Example: heat equation: disc./ in space with our favourite "1 -2 1" matrix.

We looked at eigenvalues before: they depend on h. Largest magnitude is $-4/h^2$. Need this inside the stability region.

Choose forward Euler: need $\lambda k > -2$.

Leads to restriction on k for stability in time:

$$k \leq \frac{1}{2}h^2$$

Stability in finite difference calculations

A fourth-order problem

$$u_t = -u_{xxxx}.$$

How to discretize? Think $(u_{xx})_{xx} \ldots$, this leads, eventually, to

$$v_j^{n+1} = v_j^n - \frac{k}{h^4}\left(v_{j-2}^n - 4v_{j-1}^n + 6v_j^n - 4v_{j+1}^n + v_{j+2}^n\right)$$

[demo_07_biharmonic.m]

Note ridicuously small time steps required. Let's try to see why (a stability issue) and what we can do about it (implicit A-stable ODE methods).

von Neumann analysis

One approach is *von Neumann Analysis* of the finite difference formula, also known as *discrete Fourier analysis*, invented in the late 1940s.

Suppose we have periodic boundary conditions and that at step n we have a (complex) sine wave

$$v_j^n = \exp(i\xi x_j) = \exp(i\xi jh),$$

for some wave number ξ. Higher ξ is more oscillatory. We will analysis whether this wave grows in amplitude or decays (for each ξ). For stability, we want all waves to decay.

For the biharmonic diffusion equation, we substitute this wave into the finite difference scheme above, and factor out $\exp(i\xi h)$ to get

$$v_j^{n+1} = g(\xi) v_j^n,$$

with the *amplification factor*

$$g(\xi) = 1 - \frac{k}{h^4}\left(e^{-i2\xi h} - 4e^{-i\xi h} + 6 - 4e^{i\xi h} + e^{i2\xi h}\right).$$

This can be simplified to:

$$g(\xi) = 1 - \frac{16k}{h^4}(\sin(\xi h/2))^4.$$

As ξ ranges over various values sin is bounded by 1 so we have

$$1 - 16k/h^4 \leq g(\xi) \leq 1.$$

A mode will grow if $|g(\xi)| > 1$. Thus for stability we want $|g(\xi)| \leq 1$ for all ξ , i.e.,

$$1 - 16k/h^4 \geq -1, \qquad \text{or} \qquad \boxed{k \leq h^4/8.}$$

For $h = 0.025$, as in the demo code, this gives

$$k \leq 4.883e - 08.$$

This matches our experiment convincingly, but confirms that this finite difference formula is not really practical.

Method-of-lines

As an alternative to von Neumann analysis, we follow the linear stability analysis for the ODE methods. The spatial discretization gives us (numerically anyway) the eigenvalues of the *semidiscrete* system. Need these eigenvalues to lie inside the absolute stability region of the ODE method.

Note: this involves the eigenvalues of the semidiscrete system, not the original right-hand-side of the PDE.

Demo: in Matlab, run demo_07_biharmonic, then use 'eigs' to compute 'largest magnitude' eigenvalues of the *discretized* biharmonic operator: need k times these less than 2 for forward Euler stability. Note this gives almost the same restriction as observed in practice (and calulated with von Neumann analysis)

Stability, Consistency, Convergence

Lax equivalence theorem: for linear PDEs, consistency + stability implies convergence.

With a particular notion of stability: **Lax-Richtmyer Stability**...

Suppose we have $k = 0.4h^2$ (some fixed relationship). Say we can write our fully-discrete system as

$$U^{n+1} = B(k)U^n + b^n(k)$$

where $B(k)$ is a matrix. This is the case for our heat example with "1 -2 1" and forward Euler for example.

Defn: Linear method in this form is *Lax–Richtmyer* stable if

$$||B(k)^n|| \leq C_T$$

where C_T constant indep of k, n but could depend on final time T.

Example: Fisher-KPP Equation

Independent 1937 discoveries for biological applications (spread of species): Fisher; Kolmogorov, Petrovsky, and Piscounov

$$u_t = \epsilon u_{xx} + u - u^2.$$

Solutions: traveling waves. Explicit finite difference model is similar to heat equation. Note we have "nonhomogeneous" BCs $u(0) = 1$, $u(20) = 0$; the former is implemented by using an extra vector "BC". [demo_07_fisher_kpp.m]

```
                   +--                                                            --+
                   |         |-2   1                 |  |  n |     |1/h^2|             |
                   |         |                       |  | u  |     |     |             |
                   |         | 1  -2   1             |  |  1 |     |  0  |             |
                   |         |                       |  |    |     |     |             |
 n+1    n          |         |   .   .   .           |  |    |     |  .  |          2  |
u     = u   + k *  | eps/h^2 |    .   .   .          |  |    |  +  |  .  |  +  (u -u ) |
                   |         |     .   .   .         |  |    |     |  .  |      j  j   |
                   |         |                       |  |    |     |     |             |
                   |         |       1   -2   1      |  |  n |     |     |             |
                   |         |                       |  | u  |     |     |             |
                   |         |            1   -2     |  |  N |     |  0  |             |
                   +--                                                            --+
                                                                  BC
```

Accuracy on heat equation

Forward and Backward Euler are both consistent with an expected error of $O(k) + O(h^2)$.

discuss w.r.t. stability of each scheme...

FE: $k = O(h^2)$ Maybe you only want first-order accuracy, is so, this extra work is wasteful.

(Yet another "definition" of stiffness: if your choice of timestep k is motivated by stability rather than accuracy, you are probably dealing with a stiff problem.)

Higher-order in time

Even if we want second-order perhaps there are better ways, use a better ODE solve: trapezoidal rule in time + second order in space. When used on heat equation, this is called "Crank–Nicolson":

$$v^{n+1} = v^n + \frac{k}{2} L v^{n+1} + \frac{k}{2} L v^n.$$

or

$$B v^{n+1} = A v^n$$

Note the stencil of this scheme:

```
                (j+1,n+1)
x------x------x
       |
       |
x------x------x
     (j,n)
```

Caution

Sometimes hard to tell from numerical convergence study which terms are dominating. Can also do tests to isolate the error components in h and k.