Abstract

A hybrid asymptotic-numerical approach is developed to study hotspot patterns for a three-component 1-D reaction-diffusion (RD) system that models urban crime with police intervention. Our analysis is focused on a scaling regime where there are two distinct competing mechanisms for producing complex spatio-temporal dynamics of hotspot patterns; a mechanism to annihilate hotspots and a further mechanism to nucleate new hotspots from a quiescent background. The nucleation threshold for steady-state hotspot patterns arises from a saddle-node bifurcation point of hotspot equilibria. By deriving a new analytical expression for a hotspot profile, combined with a local normal form analysis, our asymptotic analysis provides a rather accurate prediction of this nucleation threshold. From a numerical computation of the spectrum of the linearization around a two-boundary hotspot pattern, we have identified instability parameter thresholds for both zero-eigenvalue crossings and Hopf bifurcations. Overall, this provides a phase diagram in parameter space where distinct types of dynamical behaviors can occur from perturbations of this two-boundary hotspot steady-state solution. In one region of this phase diagram, corresponding to a small police diffusivity, a two-boundary hotspot steady-state is unstable to a subcritical asynchronous oscillatory instability in the hotspot amplitudes. This instability normally leads to the complete annihilation of one of the hotspots. However, in the regime where this instability is coincident with the non-existence of a one-hotspot steady-state, we show that hotspot patterns undergo complex nucleation-annihilation dynamics that are characterized by large-scale and persistent oscillations of the hotspot amplitudes. This case study suggests that there are parameter regimes in the three-component RD system where the problem of predicting where and when hotspots of crime will appear and then essentially disappear is largely intractable.

1 Introduction

Crimes due to residential burglaries are not generally uniformly distributed within cities. Instead, they are denser in some areas and sparser in others, which results in crime patterns that tend to be spatially localized in certain “hotspots” of criminal activity [1] (see also [2]). The development of agent-based and PDE continuum-based mathematical modeling approaches to predict spatial patterns of urban crime originates from the foundational studies in [3, 4, 5], which are based on postulated interactions between two primary field variables: the attractiveness of the site for burglary and the density of criminals. A key ingredient in their modeling framework is that crime hotspots can result from a “near-repeat” victimization effect, which suggests that a crime at one site can create an environment that encourages further crime events near that site within a certain time period [1].
To model the effect of police intervention on crime hotspots two distinct approaches have been used. By using agent-based simulation modeling, various detailed real-world policing strategies can and have been incorporated (cf. [6], [7]). A second direction, which is more amenable to analysis, is to formulate and analyze three-component PDE-based reaction-diffusion (RD) systems that include the police density as an extra field variable (cf. [6], [8], [9], [10], [11], [12]). An extension of this three-component system to formulate an optimal control strategy so as to minimize the overall crime rate through a policing effort that adapts to a dynamically evolving crime pattern has also been considered [9].

The goal of this article is to analyze, within a new scaling regime, some novel complex spatio-temporal hotspot patterns that occur for a three-component 1-D RD model of urban crime with police deployment. On the interval $-L < x < L$, this RD system is given in dimensionless form by

$$A_t = \varepsilon \delta^2 A_{xx} - A + \rho A + \alpha, \quad (1.1a)$$

$$\rho_t = D(\rho_x - 2\rho A_x/A) - \rho A + \gamma - \alpha - \rho U, \quad (1.1b)$$

$$\tau U_t = D(U_x - qUA_x/A)_x, \quad (1.1c)$$

where $A_x(\pm L) = \rho_x(\pm L) = U_x(\pm L) = 0$. Here $A(x,t)$ is the attractiveness of the environment to burglary, while $\rho(x,t)$ and $U(x,t)$ are the population densities of criminals and police, respectively. The constant $\alpha > 0$ is the spatially uniform baseline attractiveness, $\gamma - \alpha$ represents the constant rate at which new criminals are introduced, $D$ is the criminal diffusivity, and $D_p \equiv D/\tau$ is the police diffusivity. The chemotactic drift term $-2D(\rho A_x/A)_x$ represents the tendency of criminals to move towards sites with higher attractiveness. Likewise, the police are assumed to undergo a biased random walk toward areas of higher attractiveness, with the parameter $q > 1$ measuring the degree of focus in the police patrol towards the maxima of the attractiveness field $A$. In particular, the choice $q = 2$ is the “cops-on-the-dots” strategy (cf. [6, 9]) where the police have the same drift tendency as the criminals towards maxima of $A$. Moreover, we will assume, as in [13] and [14], that

$$\gamma > 3\alpha/2. \quad (1.2)$$

On this range of $\gamma$, the uniform steady-state solution to (1.1) in the absence of police, given by $A_e = \gamma$ and $\rho_e = 1 - \alpha/\gamma$, is unstable as $\varepsilon \to 0$ to the formation of hotspots [14]. By integrating the $U$ equation (1.1c) over the domain, we conclude that the total “mass” of police is conserved in time, and we denote this key system parameter as

$$U_0 = \int_{-L}^{L} U(x,t) \, dx. \quad (1.3)$$

To analyze (1.1) it is convenient to introduce the new variables $V$ and $u$, as introduced in [13, 14, 12], defined by

$$\rho = VA^2, \quad U = uA^q. \quad (1.4)$$

In terms of these variables, (1.1) transforms on $-L < x < L$ to

$$A_t = \varepsilon^2 A_{xx} - A + VA^3 + \alpha, \quad (1.5a)$$

$$(VA^2)_t = D(A^2V)_x - VA^3 + \gamma - \alpha - uVA^{2+q}, \quad (1.5b)$$

$$\tau(u^q)_t = D(u^q u_x)_x. \quad (1.5c)$$

Our analysis of (1.1) and (1.5) will focus on the singularly perturbed limit $\varepsilon \ll 1$, where the attractiveness field can be localized within certain hotspot regions, also referred to in this article as “spikes”. A numerically-computed two-hotspot steady-state pattern from (1.1) is illustrated in Figure [1] on the domain $[-3,3]$ with parameter values $\alpha = 1, \gamma = 2, \varepsilon = 0.03, \tau = 1, D = 4$, for the choices $q = 3$ and $q = 2$. We observe that when $q = 3$ the spatial extent of $U$ is more focused as compared to the “cops-on-the-dots” case $q = 2$. The initial conditions for generating Figure [1] were taken as random perturbations of the spatially homogeneous steady-state.
When $U_0 = 0$, (1.1) reduces to a two-component PDE system, which was first derived and analyzed in [3, 4]. This reduced system has been studied extensively. A rigorous existence theory was established in [15, 16, 17]. The computation of global snaking-type bifurcation diagrams was undertaken in [18] for the regime where $\alpha < \gamma < 3\alpha/2$. Hotspot equilibria for the regime $D \gg 1$ were constructed formally in [13] using matched asymptotic expansions and established rigorously in [19] by using a Lyapunov-Schmidt reduction. A weakly nonlinear analysis was implemented in [20] to analyze the emergence of spatial patterns near the Turing bifurcation point associated with the spatially uniform steady-state. The nucleation behavior of hotspot patterns and their slow dynamics in the regime $D = O(1)$ was analyzed in [14] in the singular limit $\varepsilon \ll 1$.

In contrast, there are relatively few studies for the analysis of spatio-temporal pattern formation for the three-component crime model with police. For the regime $D = O(\varepsilon^{-2}) \gg 1$, the existence and linear stability of hotspot steady-states for the simple police interaction model, in which $-\rho U$ in (1.1b) was replaced by $-U$, was analyzed in [11]. These results were extended in [12], where the predator-prey type police interaction as given in (1.1b) was considered for the regime $D = O(\varepsilon^{-2})$. One key result of these studies was the identification of a Hopf bifurcation for steady-state hotspot patterns that leads to asynchronous oscillations in the amplitudes of the hotspots. This initial instability effectively displaces crime between adjacent spatial regions. More recently, in [10] a weakly nonlinear analysis of the three-component RD model, using an alternative dimensionless formulation to (1.1) and in the non-singular limit, was implemented to characterize the bifurcation properties of the spatially uniform steady-state. For a certain parameter regime, this rigorous analysis established the existence of a Hopf bifurcation that initiates large-scale pattern formation and the oscillations that occur far from the spatially uniform state. Numerical computations in [10] have illustrated that, in the fully nonlinear regime, (1.1) can exhibit a wide range of highly complex spatio-temporal hotspot patterns that await a theoretical understanding.

The goal of this article is to explore the steady-state and dynamical behavior of crime hotspots that occur for (1.1) in the singular limit $\varepsilon \ll 1$, but where $D = O(1)$. By focusing on the asymptotic regime where $D = O(1)$, we can incorporate into a single scaling regime two distinct competing mechanisms that are key ingredients for producing highly complex spatio-temporal dynamics of hotspot patterns; a mechanism to annihilate hotspots and a further mechanism to nucleate new hotspots from a quiescent background. For the two-component RD urban crime model without police, it was shown in [14] that when $D = O(1)$ new hotspots of criminal activity can nucleate from a crime-free quiescent background. This nucleation behavior is a consequence of a saddle-
node bifurcation point of hotspot equilibria. For the three-component model (1.1), but in the scaling regime $D = \mathcal{O}(\varepsilon^{-2}) \gg 1$, there is no hotspot nucleation behavior. Instead, it was shown in [12] from a nonlocal eigenvalue problem (NLEP) linear stability analysis that, on a certain range of $D$, the linearization of a steady-state multispike pattern can undergo a Hopf bifurcation as $\tau$ is increased, which leads to an asynchronous temporal oscillatory instability in the hotspot amplitudes. Since the police diffusivity is $D_p = D/\tau$, this regime occurs when the police response is sufficiently “sluggish” and $D_p$ is below a threshold. Full numerical computations in [12] have suggested that this oscillatory instability is subcritical, and that it typically leads to the annihilation of certain hotspots in the pattern.

From numerical simulations of (1.1) we now illustrate a few dynamical processes that can occur for hotspot patterns of (1.1) when $\varepsilon \to 0$ and $D = \mathcal{O}(1)$. Firstly, new hotspots of criminal activity can be nucleated in low crime regions when the criminal diffusivity $D$ is decreased. This process is illustrated in the left panel of Figure 2 for an initial condition consisting of a two-hotspot steady-state solution to (1.5) when $D$ is varied in time as $D = 4 - 10^{-5} t$. As $D$ is gradually decreased, the initial pattern becomes unstable and new hotspots are nucleated at the domain boundary and in between two adjacent hotspots. For the same parameter set, but where $D$ is gradually increased in time as $D = 30 + 10^{-5} t$, in the right panel of Figure 2 we show the monotonic collapse of one hotspot in the multispike pattern as $D$ increases past a threshold $D_c$. The surviving hotspot then undergoes a slow subsequent drift towards its steady-state location. This phenomenon is referred to as competition instability. Finally, when $\tau$ is large enough and when $D < D_c$, Figure 3 shows that the two-hotspot steady-state is unstable to an asynchronous oscillatory instability of the hotspot amplitudes, which ultimately annihilates one of the hotspots in an oscillatory collapse. The surviving hotspot then drifts over a very long time period towards the domain midpoint. Movies of these three processes are given in Appendix C.

One qualitative goal of this study is to develop a hybrid asymptotic-numerical approach to identify parameter ranges where both hotspot nucleation and asynchronous amplitude oscillations can occur simultaneously in the $D = \mathcal{O}(1)$ regime for hotspot patterns of (1.1). For a specific two-boundary spike pattern, we will show that the combined effect of these two distinct mechanisms, which we refer to as nucleation-annihilation dynamics, can lead to irregular, but persistent, large scale oscillations of the hotspot amplitudes for certain time-independent parameter sets. In such a parameter regime, the problem of predicting where and when crime hotspots appear and then nearly disappear is largely intractable. Based on this detailed case study, and together with a few further numerical experiments, we conjecture that nucleation-annihilation dynamics are a key mechanism for generating complex spatio-temporal hotspot dynamics for (1.1). Overall, our proposed mechanism is reminiscent of the creation-annihilation dynamics uncovered in [21] and [22], where the merging of localized peaks together with their nucleation from a quiescent background whenever the inter-peak separation exceeded a threshold were shown to be the underlying mechanisms for highly irregular and chaotic dynamics that can occur for a chemotaxis system with logistic growth.

In §2 we use a formal singular perturbation analysis in the limit $\varepsilon \to 0$ to construct hotspot steady-state solutions to (1.5). For the steady-state problem, (1.1) reduces to a singularly perturbed nonlocal two-component BVP. Based largely on the derivation of a new explicit formula for the homoclinic connection characterizing the hotspot profile, we provide asymptotically accurate predictions for the steady-state hotspot amplitude even for only moderately small $\varepsilon$. When the police deployment $U_0$ is below a threshold, hotspot steady-state solutions are shown to exist provided that $D > D_{\text{crit},\varepsilon}$. This critical value is the minimum value of $D$ at which the outer limit of the steady-state solution exists, and is found to closely approximate a saddle-node bifurcation point for the existence of hotspot steady-states. Near this critical value of $D$, hotspots are nucleated from a quiescent background. In §3 we provide a higher order analysis, extending and improving that in [14] for the case of no police, to obtain an accurate asymptotic prediction of the hotspot nucleation threshold $D_{\text{crit},\varepsilon}$ when $U_0 \geq 0$. This analysis relies on a normal form equation derived in [23] in the study of self-replication behavior of mesa patterns. In §3 we also present global bifurcation results for hotspot equilibria as computed using the bifurcation software pde2path [24].

In §4 we study the linear stability of two specific steady-state hotspot patterns. For a one-hotspot steady-
Figure 2: Left: Hotspot nucleation process as $D$ is slowly decreased in time with $D = 4 - 10^{-5}t$; Right: A competition instability as $D$ is increased leading to the monotonic collapse of a spike with $D = 30 + 10^{-5}t$. The colormap based on the amplitude of $A$ is shown as $D$ is changed. The other parameters are $q = 3, \varepsilon = 0.05, L = 3, \alpha = 1, \gamma = 2, U_0 = 5, \tau = 1$. See Appendix C for movies of these processes.

state solution, we derive an NLEP that has three distinct nonlocal terms. From a numerical study of this NLEP we show that one-hotspot steady-states are always linearly stable. For a two-boundary hotspot steady-state solution, in §4.3 we numerically solve the eigenvalue problem for the linearization to construct a linear stability phase diagram in the $\tau$ versus $D$ parameter space, delineating ranges of parameters where there are dynamically distinct solution behaviors. Nucleation-annihilation dynamics, resulting in large-scale persistent oscillations of the hotspot amplitudes, is found to occur in one region of this phase diagram. Finally, in §5 we remark on some possible implications of our study, and we discuss a few problems that warrant further investigation.

2 The Steady State Hotspot Solution

In this section, we study the steady-state problem for (1.5) in the limit $\varepsilon \to 0$. To construct a $K$-hotspot steady-state solution with $K \geq 1$, we use the method of matched asymptotic expansions and follows the approach in [14]. That is, we first construct a single hotspot solution centered at $x = 0$ on $(-l, l)$ such that $l = L/K$. We then apply a “gluing” technique to determine $K$-spike steady-states. The steady-state problem for (1.5) on the canonical domain $|x| \leq l$, with no-flux conditions $A_x = V_x = u_x = 0$ at $x = \pm l$, is

\begin{align}
\varepsilon^2 A_{xx} - A + VA^3 + \alpha &= 0, \\
D (A^2 V)_x - VA^3 + \gamma - \alpha - uVA^{2+q} &= 0, \\
D (A^q u_x)_x &= 0.
\end{align}

(2.1)

By integrating (2.1c) with $u_x = 0$ at $x = \pm l$, we find that $u$ is a constant. By using $U = uA^q$ and (1.3), it follows that $u$ is given by

\begin{equation}
\frac{U_0}{\int_{-L}^{L} A^q \, dx} = \frac{U_0}{K \int_{-l}^{l} A^q \, dx},
\end{equation}

(2.2)

where $U_0$ is the constant total police deployment. In this way, the three-component steady-state problem (2.1)
Figure 3: Annihilation of hotspots arising from an asynchronous oscillatory instability. Left: Asynchronous oscillatory instabilities of hotspot amplitudes occur when $\tau$ exceeds a Hopf bifurcation boundary. Middle: Long time behaviour of the remaining spike. Right: The spot amplitudes of $A$ at $x = \pm 1.5$ computed numerically from the full PDE system (1.1). The parameters are $q = 3$, $\varepsilon = 0.05$, $L = 3$, $D = 60$, $\alpha = 1$, $\gamma = 2$, $U_0 = 5$, $\tau = 10$. See Appendix C for a movie of the oscillatory collapse.

This system reduces to the following two-component, but nonlocal, system:

\begin{align}
\varepsilon^2 A_{xx} - A + V A^3 + \alpha &= 0, \quad |x| \leq l; \quad A_x(\pm l) = 0, \quad (2.3a) \\
D(A^2 V_x)_x - VA^3 + \gamma - \alpha - \frac{U_0}{K \int_{-l}^{l} A^q dx} V A^{2+q} &= 0, \quad |x| \leq l; \quad V_x(\pm l) = 0. \quad (2.3b)
\end{align}

This system has a similar structure as the steady-state problem in [14] but it contains an extra nonlocal term, which makes the analysis much more intricate than in [14]. A naive asymptotic analysis would be to let $A \sim A_0/\varepsilon = O(\varepsilon^{-1})$ in the hotspot inner region, which has the effect of neglecting the $\alpha$ term in (2.3a) to leading order. Upon doing so, this makes the homoclinic solution satisfy $A_0(y) \to 0$ as $y \to \infty$, where $y \equiv x/\varepsilon$ [14]. However, this leading order analysis is not sufficiently accurate for evaluating the non-local term $\frac{U_0}{K \int_{-l}^{l} A^q dx}$.

As such, we need a new more sophisticated approach than in [14] to construct the hotspot steady-state, which is based on a more accurate calculation of the homoclinic profile. In this more refined approach, we will construct a solution with $A = O(1)$ in the inner region, but where $A$ will ultimately depend weakly on $\varepsilon$.

We first introduce the inner variable $y = x/\varepsilon$ in (2.3) to obtain on $-\infty < y < \infty$ that

\begin{align}
A_{yy} - A + V A^3 + \alpha &= 0, \quad (2.4a) \\
D(A^2 V_y)_y &= \varepsilon^2 \left( V A^3 - (\gamma - \alpha) + \frac{U_0}{K \int_{-l}^{l} A^q dx} V A^{2+q} \right). \quad (2.4b)
\end{align}

By expanding $A \sim A_0 + \varepsilon A_1 + \ldots$ and $V \sim V_0 + \varepsilon^2 V_1 + \ldots$, we obtain from (2.4) that, to leading order, $V = V_0$ is a constant to be determined, and that $A_0$ satisfies

\begin{align}
A_{0yy} - A_0 + V_0 A_0^3 + \alpha &= 0. \quad (2.5)
\end{align}
The hotspot profile $A_0$ is determined by setting

$$A_0 = \frac{w}{\sqrt{V_0}}, \quad (2.6)$$

where $w(y)$ is the homoclinic solution to

$$w_{yy} - w + w^3 + b = 0, \quad w(0) > 0, \quad w_y(0) = 0, \quad \text{where} \quad b \equiv \alpha \sqrt{V_0} > 0. \quad (2.7)$$

As shown in Appendix A, we can derive the following new explicit result regarding the existence of a homoclinic solution to (2.7):

**Lemma 1** On the range $0 \leq b < b_c \equiv \frac{2}{\sqrt{3}}$, there exists a unique homoclinic solution $w(y)$ to (2.7) given explicitly by

$$w(y) = \frac{c(y)(1 - 2w^2_\infty) + w_\infty}{1 + c(y)w_\infty}, \quad \text{where} \quad c(y) \equiv \sqrt{\frac{2}{1 - w^2_\infty} \sech(\sqrt{1 - 3w^2_\infty} y)}. \quad (2.8)$$

Here $w_\infty > 0$ is the smallest positive root of $w^3 - w + b = 0$. The homoclinic solution satisfies $w(y) \to w_\infty$ as $|y| \to \infty$. We have $b < w_\infty < 3b/2$ and that $w_\infty \to 0$ as $b \to 0$. In the limit $b \to 0$, we recover the usual leading-order profile $w = \sqrt{2} \sech(y)$ as in [14]. In terms of $w_\infty$, we have $w(0) \equiv w_m = -w_\infty + \sqrt{2}(1 - w^2_\infty)^{1/2}$.

**Proof** To show that $w_\infty$ lies on the range $b < w_\infty < 3b/2$, we use $b = \alpha \sqrt{V_0} > 0$ and set $w_\infty = nb > 0$ in $w^3 - w + b = 0$ to obtain that $n$ satisfies

$$B(n) \equiv \frac{n - 1}{n^3} = b^2 > 0. \quad (2.9)$$

For $b \to 0$, we have $n \to 1$, so that $w_\infty \sim b$. We calculate $B(1) = 0$ and $B'(n) = n^{-4}(3 - 2n)$, so that $B'(n) > 0$ on $1 \leq n < 3/2$. Since $B(3/2) = 4/27 = b_c^2$, we conclude by the monotonicity of $B(n)$ that for any $b$ in $0 < b < b_c$, the minimal root of (2.9) must satisfy $1 < n < 3/2$, which establishes that $b < w_\infty < 3b/2$. As a result, if we define $A_{0\infty} \equiv \lim_{y \to \infty} A_0(y)$, we have that $A_{0\infty}$ satisfies

$$\alpha < A_{0\infty} \equiv \frac{w_\infty}{\sqrt{V_0}} < \frac{3\alpha}{2}. \quad (2.10)$$

The derivation of the explicit formula (2.8) for the hotspot profile is given in Appendix A.

**Remark 1** In our analysis below, we will ultimately calculate a nonlinear algebraic equation for $V_0$, which will show that $V_0 \ll 1$ as $\varepsilon \to 0$ but with an intricate dependence on $\varepsilon$. This will yield that $b = \alpha \sqrt{V_0} \ll 1$ as $\varepsilon \to 0$. However, by including the effect of $b$ in the non-perturbative explicit expression (2.8) we will always obtain an accurate approximation for the homoclinic solution. In this way, we avoid having to construct additional nested inner layers as was done in §5 of [14], in the absence of police, where the analysis was instead based on perturbing around the leading order expression $w = \sqrt{2} \sech(y)$. One key finding of [14] was that a mid-inner boundary layer with $y = O(-\log \varepsilon)$ is needed to match the inner solution to an outer solution. As a consequence, the outer solution was defined on the renormalized domain $x_\varepsilon < x < l$, where $x_\varepsilon = -\varepsilon \log \varepsilon$ (see §5 of [14]). Although our explicit formula for the homoclinic avoids having to introduce nested inner layers, the fact that $b \ll 1$ indicates that we should still use the renormalized outer domain $x_\varepsilon < x < l$ in our analysis below.

We now analyze (2.3) on the renormalized outer region $-l < x < -x_\varepsilon$ and $x_\varepsilon < x < l$. Since the hotspot solution is even, we only consider the range $x_\varepsilon < x < l$. On this range, the leading order outer solution is

$$A \sim a_0(x), \quad V \sim v_0(x), \quad (2.11)$$
where \(a_0(x)\) and \(v_0(x)\) satisfy
\[
-a_0 + v_0 a_0^3 + \alpha = 0, \quad D(a_0^2 v_0) x - v_0 a_0^3 + \gamma - \alpha - \frac{U_0}{KI} v_0 a_0^{q+2} = 0. \tag{2.12}
\]

Here we have labeled \(I \equiv 2 \int_0^l A^q dx\). In order that \(a_0\) and \(v_0\) can match to the inner solution, we require that
\[
a_0(x_\varepsilon) = A_\infty = w_\infty / \sqrt{V_0}, \quad v_0(x_\varepsilon) = V_0. \tag{2.13}
\]

From Lemma [1] we observe that \(a_0(x_\varepsilon)\) satisfies \(\alpha < a_0(x_\varepsilon) < 3\alpha/2\).

By solving the first equation in (2.12) for \(v_0\) we get
\[
v_0 = g(a_0) \equiv \frac{a_0 - \alpha}{a_0^3}, \tag{2.14}
\]
so that the second equation in (2.12) yields the following boundary value problem on \(x_\varepsilon < x < l\):
\[
D(f(a_0) a_0 x) = R(a_0) \equiv a_0 - \gamma + \frac{U_0}{KI} (a_0 - \alpha) a_0^{q-1}; \quad a_0(x_\varepsilon) = \frac{w_\infty}{\sqrt{V_0}}, \quad a_0x(l) = 0. \tag{2.15}
\]

Here we have defined \(f(a_0)\) by
\[
f(a_0) \equiv \frac{3\alpha - 2a_0}{a_0^2} = a_0^2 g'(a_0). \tag{2.16}
\]

For the well-posedness of (2.15), we need \(f(a_0) > 0\) on \(x_\varepsilon < x < l\). A sufficient condition for this to hold is that \(a_0(x_\varepsilon) < a_0(x) < 3\alpha/2\).

To ensure this range for \(a_0(x)\), we will now derive a sufficient condition for \(a_0(x)\) to be monotone increasing. To do so, we need to analyze \(R(a_0)\) in (2.15). We calculate that \(R(\alpha) = \alpha - \gamma < 0\) and that
\[
R'(a_0) = 1 + \frac{U_0}{KI} q a_0^{q-2} (a_0 - \alpha + \alpha q); \quad \alpha > 0, \tag{2.17}
\]
for \(a_0 > \alpha\). Therefore, since \(\alpha < a_0(x_\varepsilon)\), we have \(R(a_0) < 0\) on the interval \(a_0(x_\varepsilon) < a_0(x) < 3\alpha/2\) when \(R(3\alpha/2) < 0\). By using the expression for \(R(a_0)\) in (2.15), we observe that \(R(3\alpha/2) < 0\) holds provided that the total police deployment \(U_0\) satisfies
\[
U_0 < U_{0,\max} \equiv 2 \left( \frac{2}{3} \right) q^{-1} (\gamma - 3\alpha/2) \alpha^q KI, \tag{2.18}
\]
where \(I = 2 \int_0^l A^q dx\). Observe that \(U_{0,\max} > 0\) since \(\gamma > 3\alpha/2\) from (1.2). The inequality (2.18) provides a maximum threshold for the total level of police deployment for which steady state hotspot solutions can exist. We will assume below that there are limited police resources so that (2.18) is satisfied.

Under the condition (2.18), which establish that \(R(a_0) < 0\) on \(a_0(x_\varepsilon) < a_0 < 3\alpha/2\), it follows that \(a_0\) is monotone increasing on \(x_\varepsilon < x < l\). To see this, we integrate (2.15) and use \(a_0x(l) = 0\) to get
\[
Df(a_0) a_0 x \bigg|_x^l = -Df(a_0) a_0 x = \int_x^l R(a_0(s)) ds.
\]

Since \(R(a_0) < 0\) and \(f(a_0) > 0\) on \(\alpha < a_0(x_\varepsilon) < a_0 < 3\alpha/2\), we conclude that \(a_0x > 0\) on this range.

Next, we reduce (2.15) to a simple quadrature by multiplying both sides of (2.15) by \(f(a_0) a_0 x\) and integrating the resulting expression using the boundary condition \(a_0x(l) = 0\). This yields that
\[
\frac{D}{2} (f(a_0) a_0 x)^2 = \int_x^l f(a_0(\eta)) R(a_0(\eta)) a_0 \eta d\eta. \tag{2.19}
\]
Upon labeling $\mu \equiv a_0(l)$, and using the monotonicity of $a_0$, we conclude that

$$\frac{D}{2}(f(a_0)a_{0x})^2 = \int_{\mu}^{\frac{a_0(x)}{a_0(x)}} f(s)R(s) ds = G(\mu; z) - G(a_0; z),$$

(2.20)

where we have defined the anti-derivative $G$, upon using (2.15) for $R$, by

$$G'(s; z) = -f(s)R(s) \equiv f(s) \left( \gamma - s - \frac{U_0}{KI}(s - \alpha)s^{q-1} \right), \quad \text{where } z \equiv \frac{U_0}{KI}. \quad (2.21)$$

After substituting $f(s)$ from (2.16) into (2.21) and integrating once, we obtain that $G(s; z)$ is given explicitly by

$$G(s; z) = 2s - (2\gamma + 3\alpha) \log(s) - \frac{3\alpha\gamma}{s} + z \left( \frac{2s^q}{q} - \frac{5\alpha s^{q-1}}{q - 1} + \begin{cases} 3\alpha^2 \log(s) & \text{if } q = 2 \\ 3\alpha^2 s^{q-2} & \text{if } q \neq 2 \end{cases} \right). \quad (2.22)$$

Since $f(a_0) > 0$ and $R(a_0) < 0$ on $a_0(x_\varepsilon) < a_0 < 3\alpha/2$, it follows that $G'(s; z) > 0$ on this range so that the inequality $G(\mu; z) - G(a_0; z) > 0$ is guaranteed. As a result, since $a_{0x} > 0$, we obtain from (2.20) that

$$f(a_0)a_{0x} = \sqrt{\frac{2}{D}}\sqrt{G(\mu; z) - G(a_0; z)}. \quad (2.23)$$

Since the ODE (2.23) is separable, we get

$$\chi(a_0(x); z) = \sqrt{\frac{2}{D}}(x - x_\varepsilon), \quad (2.24)$$

where we have defined

$$\chi(a_0(x); z) \equiv \int_{a_0(x_\varepsilon)}^{a_0(x)} \frac{f(s)}{\sqrt{G(\mu; z) - G(s; z)}} ds.$$

Then, by setting $x = l$ and $a_0(l) = \mu$ we obtain an implicit equation for $\mu \equiv a_0(l)$ given by

$$\sqrt{\frac{2}{D}}(l - x_\varepsilon) = \chi(\mu; z) = \int_{a_0(x_\varepsilon)}^{\mu} \frac{f(s)}{\sqrt{G(\mu; z) - G(s; z)}} ds, \quad (2.25)$$

where $a_0(x_\varepsilon) = w_\infty/\sqrt{V_0}$ from (2.13). The integral in (2.25) is improper at $s = \mu$. In order to obtain a more tractable formula for $\chi(\mu; z)$, we integrate the expression by parts by using $f(s) = -G'(s)/R(s)$ to obtain the proper integral

$$\chi(\mu; z) = -\frac{2\sqrt{G(\mu; z) - G(a_0(x_\varepsilon); z)}}{R(a_0(x_\varepsilon))} \int_{a_0(x_\varepsilon)}^{\mu} \frac{\sqrt{G(\mu; z) - G(s; z)}}{(R(s))^2} \left( 1 + z \left( q s^{q-1} - \alpha(q - 1)s^{q-2} \right) \right) ds, \quad (2.26)$$

where $R(s)$ is given by

$$R(s) \equiv s - \gamma + z(s - \alpha)s^{q-1}, \quad \text{with } z \equiv \frac{U_0}{KI}. \quad (2.27)$$

We observe that $\chi(a_0; z)$ is positive and monotonically increasing in $a_0$ since $\chi'(a_0; z) = \frac{f(a_0)}{\sqrt{G(\mu; z) - G(a_0; z)}} > 0$ for $U_0 < U_{0,max}$. Since $\mu = a_0(l) \leq 3\alpha/2$, the largest possible value of $\mu$ is $3\alpha/2$, and so we define $\chi_{\max}$ by

$$\chi_{\max} \equiv \chi(3\alpha/2; z). \quad (2.28)$$
Upon fixing \( l \), we conclude from \([2.25]\) that the minimum value of \( D \) for which an outer solution exists is

\[
D_{\text{crit},\varepsilon} = \frac{2(l - x_\varepsilon)^2}{\chi_{\text{max}}}^2, \quad \text{where} \quad x_\varepsilon = -\varepsilon \log \varepsilon. \tag{2.29}
\]

This asymptotic threshold suggests that as \( D \) decreases below this critical value the one-hotspot steady-state solution on the canonical domain \( |x| \leq l \) no longer exists. Our full numerical simulations below in Figure 6 show that new hotspots are created near the boundaries \( x = \pm l \), which corresponds to the midpoint between two adjacent hotspots on the original domain \(-L < x < L\). We refer to this phenomena as either hotspot insertion or nucleation. This hotspot nucleation behavior is analyzed in detail in \( \S 3 \).

Finally, we must determine an equation for \( V_0 \) by matching the behavior of \( V \) in the inner and outer regions. To do so, we first expand \( V = V_0 + \varepsilon^2 V_1 + \ldots \) in the inner region and collect \( O(\varepsilon^2) \) terms in \([2.4]\). This yields that \( V_1 \) satisfies

\[
D(A_0^2 V_{1y}) = V_0 A_0^3 - (\gamma - \alpha) + \frac{U_0}{K \int_{-l}^l A^q \, dx} V_0 A_0^{2+q}. \tag{2.30}
\]

By using the expression \( A_0 = w(y)/\sqrt{V_0} \) for the homoclinic we get

\[
\frac{D}{V_0}(w^2 V_{1y}) = H(y; z), \quad 0 < y < \infty, \tag{2.31a}
\]

where \( H(y; z) \equiv \frac{1}{\sqrt{V_0}} w^3(y) - (\gamma - \alpha) + \frac{z}{V_0^{q/2}} w^{2+q}(y) \) and \( z = \frac{U_0}{2K \int_{-l}^l A^q \, dx} \). \( \tag{2.31b} \)

In order to match to the outer solution, we must determine the far-field behavior of \( V_1 \) in \( [2.31a] \) as \( y \to \infty \). Since \( \int_0^\infty H(y; z) \, dy \) does not exist owing to the fact that \( H(\infty; z) \neq 0 \), in order to evaluate the integral of the right-hand side of \( [2.31a] \), we first rewrite it as

\[
\frac{D}{V_0}(w^2 V_{1y}) = H(y; z) - H(\infty; z) + H(\infty; z), \quad \text{where} \quad H(\infty; z) \equiv \frac{1}{\sqrt{V_0}} w_\infty^3 - (\gamma - \alpha) + \frac{z}{V_0^{q/2}} w^{2+q}_\infty. \tag{2.32}
\]

By integrating \([2.32]\), and imposing the symmetry condition \( V_{1y}(0) = 0 \), we obtain that

\[
\frac{D}{V_0} w^2 V_{1y} = \int_0^y (H(s; z) - H(\infty; z)) \, ds + y H(\infty; z). \tag{2.33}
\]

Since \( w(y) \) decays exponentially to \( w_\infty \), we conclude that \( \int_0^\infty (H(s; z) - H(\infty; z)) \, ds < \infty \). Then, by letting \( y \to \infty \) in \([2.33]\), we obtain the far field behavior

\[
V_{1y} \sim \frac{V_0}{D w_\infty^2} (c_0 + c_1 y), \tag{2.34}
\]

as \( y \to +\infty \), where the constants \( c_0 \) and \( c_1 \) are given explicitly by

\[
c_0 \equiv \int_0^\infty (H(y; z) - H(\infty; z)) \, dy = \frac{1}{\sqrt{V_0}} \int_0^\infty (w^3(y) - w_\infty^3) \, dy + \frac{z}{V_0^{q/2}} \int_0^\infty (w^{2+q}(y) - w^{2+q}_\infty) \, dy, \tag{2.35a}
\]

\[
c_1 \equiv H(\infty; z) = \frac{1}{\sqrt{V_0}} w_\infty^3 - (\gamma - \alpha) + \frac{z}{V_0^{q/2}} w^{2+q}_\infty. \tag{2.35b}
\]

Now we study the behavior as \( x \to x_\varepsilon \) of the outer solution \( v_0(x) \). From \([2.12]\), we find that \( v_0(x) \) satisfies

\[
D(a_0^2 v_0)_x = v_0 a_0^3 - (\gamma - \alpha) + z v_0 a_0^{q+2}, \tag{2.36}
\]
with \( v_0(x) = V_0 \) and \( v_0(l) = 0 \). Since \( a_0(x) = w_\infty/\sqrt{V_0} \), we obtain from (2.36) that
\[
v_{0xx}(x) \sim \frac{V_0}{Dw_\infty^2} \left( \frac{1}{\sqrt{V_0}} w_\infty^3 - (\gamma - \alpha) + \frac{z}{V_0^{q/2} w_\infty^{2+q}} \right).
\] (2.37)

Then, by expanding \( v_{0x}(x) \) in a Taylor series we get
\[
v_{0x}(x) = v_{0x}(x_\varepsilon) + (x - x_\varepsilon)v_{0xx}(x_\varepsilon) + \ldots = v_{0x}(x_\varepsilon) - x_\varepsilon v_{0xx}(x_\varepsilon) + v_{0xx}(x_\varepsilon)x + \ldots \quad \text{as} \quad x \to x_\varepsilon,
\] (2.38)
where \( x_\varepsilon \equiv -\varepsilon \log(\varepsilon) \). Now we enforce a continuity condition between (2.38) and (2.34). After using \( V_y = \varepsilon V_x \), and \( V \sim V_0 + \varepsilon^2 V_1 \), we obtain from (2.34) that
\[
v_{0x}(x) = \frac{1}{\varepsilon} V_y \sim \varepsilon V_{1y} = \varepsilon \frac{c_0 V_0}{Dw_\infty^2} + \frac{c_1 V_0}{Dw_\infty^2} x.
\] (2.39)
We observe from (2.35b) and (2.37) that the 2nd term in (2.39) matches identically with the \( v_{0xx}(x_\varepsilon) \) term in (2.38). Therefore, to ensure that the other terms agree, we must have
\[
v_{0x}(x_\varepsilon) - x_\varepsilon v_{0xx}(x_\varepsilon) = \varepsilon \frac{c_0 V_0}{Dw_\infty^2}.
\] (2.40)

Next, by using (2.14) and (2.23) we calculate
\[
v_{0x} = \frac{1}{a_0} (f(a_0) a_{0x}) = \frac{1}{a_0} \sqrt{\frac{2}{D}} \sqrt{G(\mu; z) - G(a_0; z)}.
\] (2.41)
Since \( a_0(x_\varepsilon) = A_\infty = \frac{w_\infty}{\sqrt{V_0}} \), we conclude that
\[
v_{0x}(x_\varepsilon) = \frac{V_0}{w_\infty^2} \sqrt{\frac{2}{D}} \sqrt{G(\mu; z) - G(A_\infty; z)}.
\] (2.42)

Finally, by substituting (2.35a), (2.37) and (2.42) into (2.40), we obtain that \( V_0 \) must satisfy
\[
\frac{1}{\sqrt{V_0}} \int_0^\infty (w_\infty^3 - w_\infty^3) dy + \frac{z}{V_0^{q/2}} \int_0^\infty (w_\infty^{2+q} - w_\infty^{2+q}) dy = \frac{1}{\varepsilon} \sqrt{\frac{2}{D}} \sqrt{G(\mu; z) - G(A_\infty; z)}
- \frac{x_\varepsilon}{\varepsilon} \left( \frac{1}{\sqrt{V_0}} w_\infty^3 - (\gamma - \alpha) + \frac{z}{V_0^{q/2} w_\infty^{2+q}} \right),
\] (2.43)
where \( x_\varepsilon = -\varepsilon \log(\varepsilon) \). The final step in the derivation of the equation for \( V_0 \) is to approximate the integral \( I \equiv \int_0^l A_d dx \), which determines \( z \) by \( z = U_0/(K_I) \). Since \( q > 1 \), the dominant contribution to \( I \) arises from the inner region, and we can approximate \( I \) as
\[
I \sim 2 \int_0^l (A_d - A_{\infty}) dx + 2I A_{\infty}^2 = 2 V_0^{-q/2} (\varepsilon J_q + lw_\infty^q), \quad \text{where} \quad J_q = \int_0^\infty (w^q - w_\infty^q) dy,
\] (2.44)
which yields
\[
q \sim \frac{U_0}{2K} \left( \frac{V_0^{q/2}}{\varepsilon J_q + lw_\infty^q} \right).
\] (2.45)

With this approximation, (2.43) yields a nonlinear algebraic equation for \( V_0 \)
\[
\frac{\varepsilon}{\sqrt{V_0}} J_3 + x_\varepsilon \left( \frac{w_\infty^3}{\sqrt{V_0}} - (\gamma - \alpha) + \frac{U_0}{2\varepsilon K} \frac{w_\infty^{2+q}}{J_q + lw_\infty^q/\varepsilon} \right) + \frac{U_0}{2K} \left( \frac{J_{2+q}}{J_q + lw_\infty^q/\varepsilon} \right) = \sqrt{2D} \sqrt{G(\mu; z) - G(A_\infty; z)} = 0.
\] (2.46)

In (2.46) we have defined the generic integral \( J_p \) by \( J_p \equiv \int_0^\infty (w^p(y) - w_\infty^p) dy \). From (2.46) we observe that formally \( V_0 = O(\varepsilon^2) \), but that the dependence on \( \varepsilon \) is rather intricate.
Figure 4: Comparison between numerical and asymptotic results for $\varepsilon^{-2}V_0$ (left panel) and for $A(0)$ (right panel). The solid and dashed curves correspond to the asymptotic results with $q = 3$ and $q = 2$, respectively, as computed from the nonlinear algebraic system (2.46) and (2.25). The circle and star points are obtained by numerical simulations of (1.1) with $q = 3$ and $q = 2$, respectively. The parameters are $l = 1, D = 5, \alpha = 1, \gamma = 2, U_0 = 1, K = 1$.

2.1 Numerical verification

In this subsection we verify our asymptotic results for the hotspot steady state with full numerical simulations of (1.1) by FlexPDE [25] for the choice $q = 3$ and for the “cops on the dots” case where $q = 2$. We observe that (2.46) is a highly implicit nonlinear algebraic equation for $V_0$ owing to the following features:

- $w(y)$ depends on $V_0$ through $b$ in (2.7), with the leading order behavior $w \sim \sqrt{2}\text{sech}(y)$ and $w_\infty = \lim_{y \to \infty} w(y) \sim \alpha \sqrt{v_0}$ as $V_0 \to 0$.

- $z$ as given in (2.45) depends on $V_0$ and appears in the function $G(s; z)$ of (2.22).

- $J_p \equiv \int_0^\infty (w^p(y) - w_\infty^p) \, dy$ is integrable for integers $p > 1$ (see the details for analytically calculating $J_p$ in Appendix [B]), but these integrals depend on $V_0$. Note that by using $w_\infty^3 - w_\infty + b = 0$, we can rewrite (2.7) as $w_{yy} - (w - w_\infty) + (w^3 - w_\infty^3) = 0$. As a result, by integrating over $0 < y < \infty$ we get $J_1 = J_3$.

- (2.46) contains $\mu = a_0(l)$, which is given by (2.25) and it depends on $z$ and $V_0$.

Since the equations (2.46) and (2.25) form a coupled system for the unknowns $V_0$ and $\mu$, which contains the terms $w_\infty, z, J_p$ that depend on $V_0$, we solve for $V_0$ and $\mu$ numerically using Newton’s method. In Figure 4, we compare the scaled asymptotic results $\varepsilon^{-2}V_0$ (left panel) and spike height $A(0)$ (right panel) with corresponding full simulation results of (1.1) computed using FlexPDE [25]. These plots show that for the choices $q = 3$ and $q = 2$ of police drift, the asymptotic results accurately predict the full numerical results over the full range $0 < \varepsilon < 0.05$. In contrast, in the absence of police, the asymptotic approach in [14] based on perturbing from the leading order homoclinic solution for the hotspot profile provided a substantially poorer approximation for $\varepsilon^{-2}V(0)$ when $\varepsilon \geq 0.02$ (see the right panel of Figure 4 of [14]).

To determine how the total police deployment $U_0$ affects the hot-spot steady state, in the left panel of Figure 5 we plot the spike height $A(0)$ versus $U_0$ in the range $U_0 \in [0, 4]$. This plot shows that the existence of police can effectively reduce the attractiveness of the environment to burglary. Moreover, we investigate how $l_{\text{crit}},$
defined as the critical value of the domain size for which new hotspots of criminal activity will be nucleated when $l > l_{\text{crit}}$. Qualitatively, the critical distance $2l_{\text{crit}}$ represents the maximum inter-hotspot separation that can occur before a new hotspot will be nucleated at the midpoint between adjacent hotspots. In the right panel of Figure 5 we plot $l_{\text{crit}}$ versus $U_0$, as computed from (2.29) for a fixed $D$. This plot shows, as expected intuitively, that with an increase in police deployment $U_0$ there can be a larger inter-hotspot separation where no nucleation of new hotspots of criminal activity can occur. We remark that in both panels of Figure 5, $A(0)$ and $l_{\text{crit}}$ may not exist when the police deployment $U_0$ is too large. When $U_0$ is too large, the outer solution no longer exists and steady-state hotspot patterns cannot form.

3 Global Bifurcation Structure and the Onset of Nucleation

In this section we numerically compute global bifurcation diagrams of hotspot steady-state solutions, and we provide a more refined asymptotic theory than given in §2 to analyze the onset of nucleation behavior that occurs when $l \approx l_{\text{crit},\varepsilon}$, or equivalently when $D \approx D_{\text{crit},\varepsilon}$, where $D_{\text{crit},\varepsilon}$ is given in (2.29). Recall that $D_{\text{crit},\varepsilon}$ denotes the minimum value of $D$ for which a single hotspot steady-state will exist on the domain $|x| \leq l$, while $l_{\text{crit},\varepsilon}$ is the maximum value of $l$ for which a hotspot steady-state exists when the criminals have diffusivity $D$.

3.1 Global bifurcation structure

To compute global bifurcation diagram of hotspot steady-state solutions we use FlexPDE [25] and pde2path[24,26] and we compare the full numerical results with the asymptotic result for the critical value $D_{\text{crit},\varepsilon}$ derived in (2.29). These computations give a detailed view on how new hotspots are created near the endpoints $x = \pm l$ when $D$ approaches a bifurcation point, which we denote by $D_{\text{num}}$. In Figure 6 we show the full numerical bifurcation results of (1.5) for single-hotspot steady states using the continuation software pde2path. The vertical axis on the bifurcation diagrams (a) and (c) is the boundary value $A(l)$, and the horizontal axis is $D$. Panel (b) and (d) are plots of attractiveness $A(x)$ at marked points in (a) and (c). The parameter values are $q = 3, K = 1, U_0 = 1, \gamma = 2, \alpha = 1, l = 1$, with $\varepsilon = 0.03$ in the top panels (a), (b) and $\varepsilon = 0.04$ in the bottom panels (c), (d). Both bifurcation diagrams (a) and (c) show that starting from the bottom solution branch, the steady-state solution
Figure 6: Continuation of single-hotspot steady state for $q = 3, K = 1, U_0 = 1, \gamma = 2, \alpha = 1, l = 1$. Panel (a) and (c) are bifurcation diagrams obtained by pde2path [24] with $\varepsilon = 0.03$ and $\varepsilon = 0.04$, respectively. The circle on the lower branch corresponds to the bifurcation point $D_{num}$ at which the interior-hotspot pattern becomes unstable. The cross on the lower branch corresponds to a fold point which is connected to a solution that has an interior hotspot together with a hotspot at each boundary. Panel (b) and (d) are plots of the attractiveness $A$ at the four marked points in (a), (c), respectively.
branch with one interior hotspot becomes unstable as $D$ is decreased below the circled point $D_{\text{num}}$. This branch is connected to a solution that has an interior hotspot together with a hotspot at each boundary. To show this, in panels (b) and (d) we plot the continuation of steady states by choosing a few points along the branch, where we observe that new hotspots of criminal activity can be nucleated at the domain boundary.

The global bifurcation numerical results at the circled points shown in Figure 6 are verified with a full simulation of (1.1) using FlexPDE, where we numerically compute time-dependent simulations of (1.1) as $D$ is gradually decreased until spike insertion behavior is observed. We obtain that $D_{\text{num}} = 1.493$ for $\varepsilon = 0.03$, $D_{\text{num}} = 1.434$ for $\varepsilon = 0.04$, which are exactly the circled points in Figure 6. Moreover, we compare the bifurcation points with the fold points shown in Figure 6(a), (c). This is shown in Figure 7 where the blue solid line corresponds to bifurcation points, denoted by $D_{\text{num}}$, and the purple dotted line corresponds to the fold points, denoted by $D_{\text{fold}}$. It is observed that $D_{\text{num}}$ and $D_{\text{fold}}$ approach each other as $\varepsilon \to 0$.

Next, we compare the leading order approximation $D_{\text{crit,}\varepsilon}$ derived in (2.29) with full simulation results. Recall that $\chi(\mu; z)$ reaches its maximum at $\mu = \frac{3}{2} \alpha$ which determines $\chi_{\text{max}}$ from (2.28). Moreover, it contains $z$, which relates to $V_0$. To compute $D_{\text{crit,}\varepsilon}$, we fix $\mu = \frac{3}{2} \alpha$ and compute $V_0$ and $D_{\text{crit,}\varepsilon}$ simultaneously by applying Newton’s method. The result is shown in Figure 7 where the red dashed curve is the asymptotic result $D_{\text{crit,}\varepsilon}$ obtained from (2.29) by varying $\varepsilon$ from 0.01 to 0.04, and the blue curve is obtained from full simulations using FlexPDE. From Figure 7 we observe that this leading order prediction for the critical value of $D$ is rather inaccurate unless $\varepsilon$ is very small, while for larger $\varepsilon$ it has the wrong qualitative dependence on $\varepsilon$. This motivates the need for a higher-order asymptotic theory to characterize the onset of hotspot nucleation.

To provide a more accurate analytical theory for the onset of hotspot nucleation near $x = \pm l$, we must construct a new boundary layer solution near the endpoints $x = \pm l$ when $D$ is near the critical value $D_{\text{crit,}\varepsilon}$, which was derived from the renormalized outer problem for $a_{x}(x)$ given by

$$D \left[ f(a_{x})a_{xx} \right]_x = a_x - \gamma + \frac{U_0}{Kl} (a_x - \alpha) q^{-1}, \quad x < x < l; \quad a_x(x) = \frac{w_0}{\sqrt{V_0}} a_{xx}(l) = 0,$$  

$$\varepsilon = 0.03, \quad D_{\text{num}} = 1.434 \quad \text{for} \quad \varepsilon = 0.04,$$
where \( f(a_0) \) is given in (2.16), \( I = 2 \int_0^q A^q dx \sim 2V_0^{-q/2} (\varepsilon J_q + lw_0^q) \), and \( x_\varepsilon = -\varepsilon \log(\varepsilon) \).

To illustrate how our previous analysis becomes invalid near the onset of nucleation, we determine the local behavior near \( x = l \) of the solution to (3.1) when \( D = D_{\text{crit,} \varepsilon} \), which corresponds to setting \( a_\varepsilon(l) = a_{0c} = 3\alpha/2 \). Near \( x = l \), we put \( a_\varepsilon = a_{0c} + \bar{a}(x) \), where \( \bar{a} \ll 1 \) and \( \bar{a}(l) = 0 \). Upon substituting this into (3.1), we integrate once while using \( \bar{a}(l) = 0 \) to obtain for \( x \) near \( l \) that

\[
\left( \frac{1}{2} \bar{a}^2 \right)_x \sim \beta a_{0c}^2(x - l), \quad \text{where} \quad \beta \equiv \frac{\gamma - a_{0c} - z(a_{0c} - \alpha)a_{0c}^{-1}}{2D_{\text{crit,} \varepsilon}}, \quad \text{and} \quad z = \frac{U_0}{KI}.
\] (3.2)

Upon integrating (3.2) and imposing \( \bar{a}(l) = 0 \), we obtain that \( \bar{a} \sim \sqrt{\beta}a_{0c}(x - l) \) as \( x \to l^- \). Therefore, the local behavior of the solution \( a_\varepsilon \) when \( D = D_{\text{crit,} \varepsilon} \) is

\[
a_\varepsilon(x) \sim a_{0c} + \sqrt{\beta}a_{0c}(x - l), \quad \text{as} \quad x \to l^-.
\] (3.3)

From (3.3), we conclude that when \( D = D_{\text{crit,} \varepsilon} \) and \( a_\varepsilon(l) = 3\alpha/2 \), the solution \( a_\varepsilon(x) \) no longer satisfies the required no-flux condition \( a_\varepsilon(x) = 0 \). This local analysis suggests that we need to construct a new boundary layer near \( x = l \) to characterize the onset of nucleation.

### 3.2 The solution structure near the onset of nucleation

In this subsection, we construct a new boundary layer solution that characterizes the onset of hotspot nucleation that occurs near the endpoints \( x = \pm l \) when \( D > D_{\text{crit,} \varepsilon} \). The overall analysis is similar to that done in [13] for the two-component urban crime model in the absence of police. In particular, we derive a normal form equation that has the same qualitative structure as that derived in [14], [23], but where the improved prediction for the nucleation threshold value of \( D \) now depends on the police deployment \( U_0 \).

To analyze the onset of nucleation, we first write the outer problem for the steady-state system (2.3) on the domain \( x_\varepsilon < x < l \) in the form

\[
\varepsilon^2 A_{xx} + A^3(V - g(A)) = 0,
\] (3.4a)

\[
D [A^2 V_x]_x - A + \gamma - A^3(V - g(A)) - zV A^{2+q} = 0,
\] (3.4b)

where \( g(a) = (a - \alpha)/a^3 \), \( z = U_0/(KI) \) with \( I = \int_0^q A^q dx \). We let \( a_\varepsilon(x) \) and \( D_{\text{crit,} \varepsilon} \) denote the solution to the renormalized outer problem (3.1) at the critical value where \( a_\varepsilon(l) = 3\alpha/2 \) and \( v_\varepsilon(x) = g(a_\varepsilon(x)) \). In the outer region, defined away from both the hotspot core and a new thin boundary layer to be constructed near \( x = l \), we expand the outer solution as

\[
A = a_\varepsilon + \nu a_{\varepsilon,1} + \ldots, \quad V = v_\varepsilon + \nu v_{\varepsilon,1} + \ldots, \quad \Lambda \equiv \frac{1}{\tilde{D}} = \Lambda_\varepsilon + \nu \Lambda_{\varepsilon,1} + \ldots,
\] (3.5)

where the gauge function \( \nu \ll 1 \) and the constant \( \Lambda_{\varepsilon,1} \) are to be determined. By expanding \( D = D_{\text{crit,} \varepsilon} + \nu D_{\varepsilon,1} + \ldots \) and comparing it with the expansion of \( \Lambda \), we conclude that

\[
D_{\text{crit,} \varepsilon} = \frac{1}{\Lambda_\varepsilon}, \quad D_{\varepsilon,1} = -\frac{\Lambda_{\varepsilon,1}}{\Lambda_\varepsilon^2}.
\] (3.6)

One of our goals in the analysis below is to compute the correction term \( D_{\varepsilon,1} \) for the critical value of the diffusivity threshold.

To do so, we first substitute (3.5) into (3.4) and collect powers of \( \nu \). Assuming that \( \nu \gg O(\varepsilon^2) \), we obtain that the leading order term \( a_\varepsilon \) satisfies the renormalized problem (3.1), while at the next order \( a_{\varepsilon,1} \) satisfies

\[
[f(a_\varepsilon)a_{\varepsilon,1}]_{xx} - \Lambda_\varepsilon a_{\varepsilon,1} \left( 1 + z \left(g(a_\varepsilon)a_\varepsilon^{2+q}\right)_{a_\varepsilon} \right) = \Lambda_{\varepsilon,1} \left( a_\varepsilon - \gamma + z(a_\varepsilon - \alpha)a_\varepsilon^{-1} \right),
\] (3.7)
on the domain $x_\varepsilon < x < l$ with $a_\varepsilon,1(x_\varepsilon) = 0$ and where, from (2.16), we have $f(a_\varepsilon) \equiv (3\alpha - 2a_\varepsilon)/a_\varepsilon^2$. Our local analysis below will provide the required asymptotic boundary condition for $a_\varepsilon,1$ as $x \to l$.

Near $x = l$, we construct a thin boundary layer by introducing the new variables $A_1, V_1$, and $s$ by

$$A = a_0c + \delta A_1(s) + \ldots, \quad V = v_0c + \delta^2 V_1(s) + \ldots, \quad s \equiv \frac{l - x}{\sigma},$$

where $a_0c \equiv 3\alpha/2$ and $v_0c \equiv g(a_0c) = 4/(27\alpha^2)$. The gauge functions $\delta \ll 1$ and $\sigma \ll 1$ are to be determined. The choice of different scales for $A$ and $V$ is due to the fact that $g'(a_0c) = 0$. We substitute (3.8) into (3.4) and perform a Taylor expansion of $g(A)$ around $a_0c$ to obtain that

$$\frac{\varepsilon^2 \delta}{\sigma^2} A_{1ss} + \delta^2 a_0c^3 \left[ V_1 - \frac{1}{2} A_1^2 g''(a_0c) \right] \ldots = 0, \quad \delta^2 \frac{a_0c^2}{\sigma^2} V_{1ss} = \Lambda_\varepsilon \left[ a_0c - \gamma + z v_0c a_0c^{2+q} \right] \ldots,$$

where from (2.14) we compute $g''(a_0c) = -2a_0c^{-1}$, with $a_0c = 3\alpha/2$.

To balance the terms in (3.9), we must choose the scales $\delta = \sigma = \varepsilon^{2/3}$. Then, upon using $v_0c = g(a_0c) = (a_0c - \alpha)/a_0c^3$, we obtain to leading order that (3.9) becomes

$$A_{1ss} + \left[ V_1 - \frac{1}{2} A_1^2 g''(a_0c) \right] = 0, \quad V_{1ss} = -\frac{2\beta}{a_0c}, \quad \text{where} \quad \beta = \frac{\Lambda_\varepsilon \left( \gamma - a_0c - z(a_0c - \alpha)a_0c^{-1} \right)}{2}.$$

(3.10)

Since $A$ and $V$ have no-flux boundary conditions at $x = l$, we must impose that $A_1s(0) = V_1s(0) = 0$. By solving for $V_1$ in (3.10) with $V_1s(0) = 0$, we obtain that

$$V_1 = V_{10} - \beta \frac{\varepsilon^2}{a_0c},$$

(3.11)

where $V_{10}$ is an arbitrary constant. Upon substituting $V_1$ into the $A_1$ equation of (3.10), we get

$$A_{1ss} + \left( \frac{A_1^2}{a_0c} - \frac{\beta}{a_0c} s^2 + V_{10} \right) = 0, \quad 0 < s < \infty.$$

(3.12)

To obtain the core problem, as introduced in [23], we simply rescale $s$ and $A_1$ by

$$A_1 = bU, \quad s = \xi y, \quad \text{where} \quad b = a_0c^{1/3}, \quad \xi = \beta^{-1/6}.$$

(3.13)

In this way, (3.12) transforms to the core problem [23]:

$$U_{yy} + U^2 - y^2 + k = 0, \quad 0 < y < \infty; \quad U_y(0) = 0,$$

(3.14)

where $k$ is defined by

$$k \equiv a_0c^{-2/3} V_{10}.$$

(3.15)

In terms of the original variables, we obtain from (3.8), and (3.13), that the boundary layer solution near $x = l$, characterizing the onset of the nucleation of a hotspot, is given by the local solution

$$A \sim a_0c + \varepsilon^{2/3} a_0c^{1/3} U(y), \quad V \sim v_0c + \varepsilon^{4/3} \frac{\beta^2}{a_0c^2} (k - y^2), \quad y \equiv \beta^{1/6} \varepsilon^{-2/3} (l - x),$$

(3.16)

where $a_0c = 3\alpha/2$, $v_0c = g(a_0c) = 4/(27\alpha^2)$, and $\beta$ is given in (3.10). In terms of the outer coordinate, we have

$$A_x = -a_0c^{1/2} U_y.$$

(3.17)

Therefore, upon matching to the local behavior as $x \to l$ of the outer solution $a_\varepsilon(x)$, as given by (3.3), we must have $U_y \to -1$ as $y \to \infty$.

We observe that $-U$ is the solution to the equation (3.4) in [23], where its solution behavior was established rigorously in Theorem 2 of [23]. Here we simply re-state the result regarding the core problem for the convenience of the reader.
Theorem 1 (From \cite{23}) In the limit \( k \to -\infty \), (3.14) admits exactly two solutions \( U = U^\pm(y) \) with \( U' > 0 \) for \( y > 0 \), with the following uniform expansions:

\[
U^+ \sim -\sqrt{y^2 - k}, \quad U^+(0) \sim -\sqrt{-k},
\]
\[
U^- \sim -\sqrt{y^2 - k} \left( 1 - 3 \cosh^2 \left( \frac{\sqrt{-k} y}{2} \right) \right), \quad U^-(0) \sim 2\sqrt{-k}.
\]

These two solutions are connected. For any such solution, let \( \Lambda \equiv \Lambda_k \). We observe that (3.23) reduces to equation (5.51) of \cite{14} when there is no police, i.e. when \( z = 0 \). Then, to determine \( \Lambda \), with the following uniform expansions:

\[
\tilde{a}_{\varepsilon,1}^2 = f(a_{\varepsilon}) \tilde{a}_{\varepsilon,1} \sim \left( f(a_{\varepsilon}) + f'(a_{\varepsilon})a_{\varepsilon}^{3/2}(x - l) \right) \left( \frac{a_{\varepsilon}k^{1/6}}{2} \right) \frac{1}{l - x} = \beta^{2/3}k.
\]

By substituting \( a_{\varepsilon,1} \) into (3.16) and matching with the outer solution \( a_{\varepsilon,1}(x) \) given in (3.7), we conclude that

\[
\nu \equiv \varepsilon^{4/3}, \quad \text{and} \quad a_{\varepsilon,1} \sim \left( \frac{a_{\varepsilon}k^{1/6}}{2} \right) \frac{1}{l - x}, \quad \text{as} \quad x \to l^-.
\]

To solve (3.7) subject to (3.21), it is convenient to introduce the new variable \( \tilde{a}_{\varepsilon,1} \) defined by \( \tilde{a}_{\varepsilon,1} \equiv f(a_{\varepsilon})a_{\varepsilon,1} \). Therefore, as \( x \to l^- \), we have upon using \( f(a_{\varepsilon}) = 0 \), \( f'(a_{\varepsilon}) = -2/a_{\varepsilon}^2 \), and (3.3) that

\[
\tilde{a}_{\varepsilon,1} = f(a_{\varepsilon})a_{\varepsilon,1} \sim \left( f(a_{\varepsilon}) + f'(a_{\varepsilon})a_{\varepsilon}^{3/2}(x - l) \right) \left( \frac{a_{\varepsilon}k^{1/6}}{2} \right) \frac{1}{l - x} = \beta^{2/3}k.
\]

In this way, we conclude by setting \( \tilde{a}_{\varepsilon,1} \equiv f(a_{\varepsilon})a_{\varepsilon,1} \) in (3.7), and using (3.22), that \( \tilde{a}_{\varepsilon,1} \) satisfies

\[
\left( \frac{\Lambda}{f(a_{\varepsilon})} \right) \left( 1 + z \left( g(a_{\varepsilon})a_{\varepsilon}^{2+\gamma} \right) a_{\varepsilon}^2 \right) \tilde{a}_{\varepsilon,1} = \Lambda_{\varepsilon,1} \left( a_{\varepsilon} - \gamma + z(a_{\varepsilon} - \alpha)a_{\varepsilon}^{q-1} \right), \quad x_{\varepsilon} < x < l,
\]
\[
\tilde{a}_{\varepsilon,1}(x_{\varepsilon}) = 0, \quad \tilde{a}_{\varepsilon,1} \sim \beta^{2/3}k \quad \text{as} \quad x \to l^-.
\]

We observe that (3.23) reduces to equation (5.51) of \cite{14} when there is no police, i.e. when \( z = 0 \). Then, to determine \( \Lambda_{\varepsilon,1} \) we introduce the new variable \( H \) by \( \tilde{a}_{\varepsilon,1} \equiv \Lambda_{\varepsilon,1}H \), so that \( H \) satisfies

\[
H_{xx} - \frac{\Lambda}{f(a_{\varepsilon})} \left( 1 + z \left( g(a_{\varepsilon})a_{\varepsilon}^{2+\gamma} \right) a_{\varepsilon}^2 \right) H = a_{\varepsilon} - \gamma + z(a_{\varepsilon} - \alpha)a_{\varepsilon}^{q-1}, \quad x_{\varepsilon} < x < l,
\]

where \( H(x_{\varepsilon}) = 0 \) and \( H \) is bounded as \( x \to l^- \) in the sense that \( H_l \equiv \lim_{x \to l^-} H(x) \neq 0 \). Here \( a_{\varepsilon}(x) \) is the solution to (3.1) when \( D = D_{\text{crit, } \varepsilon} \). In terms of the solution to (3.24) and the constant \( H_l \), we identify that

\[
\Lambda_{\varepsilon,1} = \frac{\beta^{2/3}k}{H_l}.
\]
Finally, by using the expression for $\beta$ in (3.10), we obtain from (3.6), together with setting $x = l$ in (3.16), that the curve $A(l)$ versus $D$ near the onset of nucleation can be parameterized as

$$D \sim D_{\text{crit,}\varepsilon} - \varepsilon^4/3 D_{\text{crit,}\varepsilon}^{4/3} \left( \frac{\gamma - a_0\varepsilon - z(a_0\varepsilon - \alpha)a_0^{-1}}{2} \right)^{2/3} \frac{k}{H_I},$$

(3.26a)

and

$$A(l) \sim a_0 + \varepsilon^{2/3} a_0^{-1/3} \left( \frac{\gamma - a_0\varepsilon - z(a_0\varepsilon - \alpha)a_0^{-1}}{2} \right)^{1/3} U(0),$$

(3.26b)

where $a_0 = 3\alpha/2$. The parameterization of this curve is in terms of the parameter $k$, as defined in the normal form (3.14). By solving (3.14) numerically in the left panel of Figure 8 we plot $U(0)$ in (3.26b) in terms of $k$.

Figure 8: Left: Plot of the bifurcation diagram of $k$ versus $U(0)$ as computed numerically from (3.14). Right: The asymptotic result (dashed curve) for $A(l)$ versus $D$ very close to the fold point, as obtained from (3.26) with $H_I = 0.0583$, is compared with the corresponding full numerical result (solid curve) computed using $pde2path$. The parameter values are $\gamma = 2, \alpha = 1, \varepsilon = 0.01, q = 3, l = 1, K = 1$.

The final step for implementing the parameterization (3.26) is to numerically compute the constant $H_I$ from the solution to (3.24). To do so, we first need to formulate an appropriate asymptotic boundary condition for $H$ that is to hold as $x \to l^-$. This is done by differentiating $\tilde{a}_{\varepsilon,1} = f(a_\varepsilon) a_{\varepsilon,1}$, while using (3.3) together with the local behavior for $a_{\varepsilon,1}$ in (3.21). Upon Taylor expanding, and recalling that $f(a_0\varepsilon) = 0$, we obtain for $x \to l^-$ that

$$\frac{d\tilde{a}_{\varepsilon,1}}{dx} = f'(a_\varepsilon) \frac{da_\varepsilon}{dx} a_{\varepsilon,1} + f(a_\varepsilon) \frac{da_{\varepsilon,1}}{dx},$$

$$\sim \left( \frac{a_0 k \beta^{1/6}}{2} \right) \left[ f'(a_\varepsilon) \sqrt{\beta} a_0 + f(a_\varepsilon) \frac{f'(a_\varepsilon)}{(l-x)^2} \right],$$

$$\sim \left( \frac{a_0 k \beta^{1/6}}{2} \right) \left[ f'(a_0) + a_0 \sqrt{\beta} f''(a_0) \right] \sqrt{\beta} a_0 \frac{\sqrt{\beta} a_0}{(l-x)} - f'(a_0) \frac{f''(a_0)}{(l-x)^2}.$$

where $a_0 = 3\alpha/2$. The parameter values are $\gamma = 2, \alpha = 1, \varepsilon = 0.01, q = 3, l = 1, K = 1$. 19
Since $f''(a_0) = 8a_0^{-3}$, we conclude that
\[
\frac{da_{\varepsilon,1}}{dx} \sim -4\beta^{7/6}k, \quad \text{as } x \to l^-.
\] (3.27)

Finally, we use $a_{\varepsilon,1} = \Lambda_{\varepsilon,1}H$, together with (3.25), to obtain the required asymptotic boundary condition
\[
H'(x) \sim -4\sqrt{\beta}H(x), \quad \text{as } x \to l^-, \quad \text{for } x = l - \delta, \text{ where } 0 < \delta \ll 1.
\] (3.28)

where we identify that $H_l = H(l)$. Here $\beta$ is given in (3.10).

To determine $H_l$, we use a shooting method on (3.24), which is iterating on the initial condition $H_0 \equiv H_0(x_\varepsilon)$ and then imposing (3.28) at $x = l - \delta$, where $0 < \delta \ll 1$. The function $a_{\varepsilon}(x)$ is solved numerically from (2.23) using Matlab ODE113. In Table 1 we computed $H_l$ for $\delta = 0.004, \gamma = 2, \alpha = 1, l = 1, q = 3$ with respect to different $\varepsilon$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>0.01</th>
<th>0.012</th>
<th>0.014</th>
<th>0.016</th>
<th>0.018</th>
<th>0.02</th>
<th>0.022</th>
<th>0.024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_l$</td>
<td>0.0538</td>
<td>0.0521</td>
<td>0.0506</td>
<td>0.0492</td>
<td>0.0480</td>
<td>0.0469</td>
<td>0.0459</td>
<td>0.0449</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>0.026</th>
<th>0.028</th>
<th>0.03</th>
<th>0.032</th>
<th>0.034</th>
<th>0.036</th>
<th>0.038</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_l$</td>
<td>0.0440</td>
<td>0.0432</td>
<td>0.0424</td>
<td>0.0417</td>
<td>0.0410</td>
<td>0.0404</td>
<td>0.0398</td>
<td>0.0392</td>
</tr>
</tbody>
</table>

Table 1: Calculation of $H_l$ for different $\varepsilon$ using shooting method on (3.24) and (3.28). The parameter values are: $\delta = 0.004, \gamma = 2, \alpha = 1, l = 1, q = 3$.

The numerical computations from Table 1 show that $H_l$ is positive, therefore, from (3.26) it can be concluded that the minimum value of $D$ which corresponds to a fold-point bifurcation that is below $D_{\text{crit,}\varepsilon}$ calculated from (2.29). Observe that $D$ attains its minimum value $D_{\text{min},\varepsilon}$ when $k = k_{\text{max}} \approx 1.46638$. Substituting $H_l$ in Table 1 into (3.26), we get an improved approximation of the critical value of $D$ at which hotspot insertion occurs. This is shown in Figure 7 where the yellow curve corresponds to $D_{\text{min},\varepsilon}$ obtained from (3.26). In the right panel of Figure 8 we show a decent comparison, in a zoomed plot, of the local approximation (3.26b) of $A(l)$ versus $D$ with full numerical results computed using pde2path.

4 Linear Stability Analysis of Steady State Hotspot Patterns

In this section we study the linear stability of steady-state hotspot solutions.

4.1 The NLEP for a single-hotspot solution

We first consider a single-hotspot steady-state solution on $|x| \leq l$ and we derive a nonlocal eigenvalue problem (NLEP) that characterizes its stability properties.

To formulate the NLEP we linearize around the steady state, denoted here by $(A_e, V_e, u_e)$, by introducing
\[
A = A_e + e^{\lambda \phi}, \quad V = V_e + e^{\lambda \psi}, \quad u = u_e + e^{\lambda \eta},
\] (4.1)

where $|\phi|, |\psi|, |\eta| \ll 1$. Upon substituting (4.1) into (4.5) we obtain that
\[
\lambda \phi = \varepsilon^2 \phi_{xx} - \phi + 3V_e A_e^2 \phi + A_e^3 \psi,
\] (4.2a)
\[
\lambda (2A_e V_e \phi + A_e^2 \psi) = D \left(2A_e V_e \phi + A_e^2 \psi_x \right)_x - (3A_e^2 V_e + (2 + q)u_e V_e A_e^{q+1}) \phi
\]
\[
- (A_e^2 + u_e A_e^{2+q}) \psi - V_e A_e^{2+q} \eta,
\] (4.2b)
\[
\tau \lambda q A_e^{q-1} u_e \phi + A_e^q \eta = D \left(q A_e^{q-1} u_e \phi + A_e^q \eta_x \right)_x,
\] (4.2c)
with Neumann boundary conditions $\phi_x = \psi_x = \eta_x = 0$ at $x = \pm l$. For $K = 1$, we recall from (2.2) and (2.45) that $u_e$ is a constant, which for $\varepsilon \ll 1$ is given by

$$u_e = \frac{U_0}{\int_{-l}^{l} A^\varepsilon dx} \sim U_0 \left( \frac{V_0^q/2}{\varepsilon \int_{-\infty}^{\infty} (w^q - w^q_\infty) dy + 2 lw^q_\infty} \right). \quad (4.3)$$

The asymptotic analysis to study (4.2) as $\varepsilon \to 0$ proceeds as follows. In the inner region we introduce the inner variable $y = x/\varepsilon$, where the inner solutions for $A_e$ and $V_e$ are $A_e \sim w(y)/\sqrt{V_0}$ and $V_e \sim V_0$. Then, to leading order, we obtain from (4.2b) and (4.2c) that $\psi_{yy} = 0$ and $\eta_{yy} = 0$, which have the bounded solutions $\eta \sim \eta_0(\equiv \eta_0)$ and $\psi \sim \psi(\equiv \psi_0)$, where $\psi_0$ and $\eta_0$ are to be determined. Moreover, from (4.2a), we get that $\Phi(y) = \phi(\varepsilon y)$ satisfies

$$\Phi'' - \Phi + 3w^2\Phi + \frac{w^3}{V_0^{3/2}} \psi_0 = \lambda \Phi, \quad -\infty < y < \infty; \quad \Phi \to 0 \quad \text{as} \quad |y| \to \infty. \quad (4.4)$$

The goal of the calculation below is to determine $\psi_0$, which will yield the NLEP from (4.4).

To do so, we first integrate (4.2c) over $|x| \leq l$ and, by imposing $\eta_x(\pm l) = u_{ex}(\pm l) = 0$, we obtain that

$$q u_e \int_{-l}^{l} A_e^{-1} \phi dx = - \int_{-l}^{l} A_e^{-1} \eta dx. \quad (4.5)$$

Since $A_e = w/\sqrt{V_0}$ in the inner region where $w = \mathcal{O}(1)$ and $V_0 \ll 1$ while $A_e = a(x) = \mathcal{O}(1)$ in the outer region, the main contribution for the integrals come from the inner region and we approximate the integrals in (4.5) only over the hotspot region. Moreover, since $\int_{-\infty}^{\infty} w^p dy$ does not converge due to the fact that $\lim_{|y| \to \infty} w = w_\infty \neq 0$, we must use replace these integrals by the finite integral $\int_{-\infty}^{\infty} (w^p - w^p_\infty) dy$. In this way, to evaluate the integral on the right hand side of (4.5), we first subtract and then add $w_\infty$ to get

$$\int_{-\delta}^{\delta} w^p dx = \int_{-\delta}^{\delta} (w^p - w^p_\infty + w^p_\infty) dx = \varepsilon \int_{-\infty}^{\infty} (w^p - w^p_\infty) dy + 2 \delta w^p_\infty \sim \varepsilon \int_{-\infty}^{\infty} (w^p - w^p_\infty) dy,$$

where $\delta w^p_\infty$ is ignored as $w_\infty \sim \alpha \sqrt{V_0} = \mathcal{O}(\varepsilon) \ll 1$. As a result, (4.5) becomes

$$\frac{q u_e}{V_0^{(q-1)/2}} \int_{-\infty}^{\infty} w^q \Phi dx \sim - \frac{\eta_0}{V_0^{q/2}} \int_{-\infty}^{\infty} (w^q - w^q_\infty) dy. \quad (4.6)$$

Upon defining the integrals $J_p$ and $I_p$ by

$$J_p \equiv \int_{-\infty}^{\infty} (w^p - w^p_\infty) dy, \quad I_p \equiv \int_{-\infty}^{\infty} w^p \Phi dy, \quad (4.7)$$

we obtain from (4.6) that

$$\eta_0 \sim - q U_e V_0^{1/2} I_{q-1}/J_q. \quad (4.8)$$

Here for convenience $J_p$ is re-defined as twice the integral used in the steady-state theory of (2) and Appendix B.

Next, we integrate the $\psi$-equation (1.2) over $x \in (l, l)$ where we impose $\psi_x(\pm l) = 0$. By using only the inner solutions to estimate the integrals, we obtain to leading order that

$$3 I_2 + \frac{\psi_0}{V_0^{3/2}} J_3 + \frac{(2 + q)}{V_0^{(q-1)/2}} u_e I_{q+1} + \frac{\eta_0}{V_0^{q/2}} J_{q+2} + \frac{\psi_0}{V_0^{1+q/2}} u_e J_{2+q} = - \lambda_0 \left( 2 V_0^{1/2} I_1 + \frac{\psi_0}{V_0} J_2 \right). \quad (4.9)$$

21
Upon using (4.8) for \( \eta_0 \) and (4.3) for \( u_e \), we obtain that (4.9) simplifies to

\[
\frac{\psi_0}{V_0^2} \left( J_3 + \varepsilon^{-1} \sqrt{V_0 U_0} J_{J_2} - J_q \right) = -3J_2 - (2 + q)\varepsilon^{-1} \sqrt{V_0 U_0} \frac{I_q}{J_q} - 2\lambda V_0^{1/2} J_1 - \frac{\eta_0}{V_0^{1/2} J_q}. \tag{4.10}
\]

By dividing this expression by \( J_3 \), we obtain that

\[
\frac{\psi_0}{V_0^2} (1 + K_q + \lambda B) = - \left( \frac{3J_2}{J_3} + (2 + q)K_q \frac{I_{J_2}}{J_{J_2}} + 2\lambda B \frac{J_2}{J_q} - qK_q \frac{I_{J_q}}{J_q} \right), \tag{4.11}
\]

where we have defined \( B \) and \( K_q \) by

\[
B \equiv V_0^{1/2} J_2 \frac{J_3}{J_q}, \quad K_q \equiv \frac{V_0^{1/2} U_0 J_{J_2}}{\varepsilon J_3} \frac{J_3}{J_q}. \tag{4.12}
\]

Finally, by substituting (4.11) into (4.4) and recalling the definition of the nonlocal term \( I_p \) in (4.7), we obtain the NLEP

\[
L_0 \Phi - \frac{w^3}{1 + K_q + \lambda B} \left( 3 \int \frac{w^2 \Phi}{J_3} \, dy + K_q \left( 2 + q \right) \int \frac{w^{1+q} \Phi}{J_{J_2}} \, dy - qK_q \int \frac{w^{q-1} \Phi}{J_q} \, dy + 2\lambda B \int \frac{w \Phi}{J_2} \, dy \right) = \lambda \Phi, \tag{4.13}
\]

where \( L_0 \Phi \equiv \Phi'' - \Phi + 3w^2\Phi \) and the integrals are over \(-\infty < y < \infty\). We remark that in this NLEP there are, for \( q = 2 \) and \( q = 3 \), three distinct nonlocal terms. Moreover, since \( V_0 = O(\varepsilon^2) \), we observe that we can treat \( K_q \) in (4.12) as an \( O(1) \) parameter whose magnitude is proportional to the police deployment \( U_0 \).

### 4.2 Numerical computations

Since the NLEP (4.13) is too complicated to analyze, we will determine the spectrum of the NLEP by using a finite difference approach. This leads to a nonlinear matrix eigenvalue problem in \( \lambda \), where we seek to determine the eigenvalue with the largest real part using an iterative approach.

To formulate the numerical problem, we re-write (4.13) as

\[
L_0 \Phi - w^3 \left( a(\lambda) \int w^2 \Phi + b(\lambda) \int w^{1+q} \Phi + c(\lambda) \int w^{q-1} + d(\lambda) \int w \Phi \right) = \lambda \Phi, \tag{4.14}
\]

where \( a, b, c, d \) are defined by

\[
a(\lambda) = \frac{3}{J_3(1 + K_q + \lambda B)}, \quad b(\lambda) = \frac{K_q(2 + q)}{J_2(1 + K_q + \lambda B)}, \quad c(\lambda) = -\frac{qK_q}{J_q(1 + K_q + \lambda B)}, \quad d = \frac{2\lambda B}{J_2(1 + K_q + \lambda B)}.
\]

Since we are interested only in even solutions, we consider (4.14) on \((0, \infty)\). Since \( w(y) \) decays exponentially to \( w_\infty \), we truncate the positive half-line to the large interval \( y \in (0, L) \), where we chose \( L = 20 \). We discretize \( \Phi(y_j) \sim \Phi_j \), where \( y_j = j \Delta y \), for \( j = 0, \ldots, N - 1 \) and \( \Delta y = L/N - 1 \), with \( N = 300 \). We use standard finite differences to approximate \( \Phi'' \), while the midpoint rule is used to approximate the nonlocal integrals in (4.14). In this quadrature, the explicit form for the homoclinic \( w \), as given in (2.8) of [2] was used. Recall that the integrals \( J_2, J_3, \) and \( J_4 \) can be calculated as in Appendix [B].

With this discretization of the NLEP, we obtain the matrix eigenvalue problem \( \mathcal{M}(\lambda) \Phi = \lambda \Phi \), where \( \Phi \equiv (\Phi_1, \ldots, \Phi_N)^T \). As the entries in \( \mathcal{M} \) depend on \( \lambda \), we used an iterative approach for determining \( \lambda \). Starting with some initial guess \( \lambda_0 \), we solve the standard matrix eigenvalue problem \( \mathcal{M}(\lambda_0) \Phi = \lambda_1 \Phi \) for the eigenvalue \( \lambda_1 \).
Figure 9: Numerical approximation of the principal eigenvalue of (4.14) for \( q = 2 \) and \( q = 3 \), with \( K_q \in (0,10) \). The parameter values are \( \gamma = 2 \), \( \alpha = 1 \), \( D = 5 \), \( \varepsilon = 0.02 \). Then from (2.46) we obtain \( V_0 = 0.0078 \) for \( q = 2 \) and \( V_0 = 0.0137 \) for \( q = 3 \). We observe that \( \lambda \) is real, with \( \lambda < 0 \) for both \( q = 2 \) and \( q = 3 \), with the largest real part. Further iterates are computed from the recursion \( \mathcal{M}(\lambda_i)\Phi = \lambda_{i+1}\Phi \). Numerically, we found that this iterative approach converged rather rapidly to a real-valued eigenvalue for both \( q = 2 \) and \( q = 3 \), and as \( K_q \) was varied.

In terms of the parameter \( K_q \), our numerical approximation of the principal eigenvalue of (4.14) is plotted for \( q = 2 \) and \( q = 3 \) in Figure 9. The results shown in Figure 9 suggest that (4.14) has no unstable eigenvalues for any \( K_q \geq 0 \). As a result, we conclude that a single hotspot steady-state solution is linearly stable.

4.3 Complex spatio-temporal dynamics of multispike patterns

In this section, we compute the competition instability threshold \( D_c \) of the diffusivity \( D \) that triggers the collapse of a spike in a specific multispike pattern. This occurs when an eigenvalue crosses above zero for the spectral problem (4.2) as the diffusivity \( D \) is increased. Threshold values for an oscillatory instability in the amplitudes of the spikes are also computed numerically as \( \tau \) is varied. These results for instability thresholds are then displayed in a phase diagram in the \( \tau \) versus \( D \) parameter space. We remark that the vertical \( \tau \) axis in this phase diagram determines the police diffusivity \( D_p \) via \( D_p = D/\tau \). For a specific multispike pattern we will show that complex spatio-temporal hotspot dynamics can occur in a certain region of this parameter space.

Owing to the fact that the outer approximations for \( \psi \) and \( \eta \) in (4.2) satisfy spatially heterogeneous BVPs, it is rather intractable analytically to derive an NLEP for studying the linear stability of multispike equilibria. As such, we must proceed with a direct numerical approximation of the eigenvalue problem (4.2).

To approximate the discrete eigenvalues of (4.2), we first compute the steady-state \( \bar{A}_e, \bar{V}_e, \bar{u}_e \) and their derivatives. To do so, for the full system (1.1) we use the finite difference method and discretize \( A(x_j) \sim \bar{A}_j \), \( \rho(x_j) \sim \rho_j \), and \( U(x_j) \sim \bar{U}_j \), where \( x_j = j\Delta x \) for \( j = 1, ..., N \) and \( \Delta x = 2L/(N-1) \). In this way, we obtain \( N \)-dimensional steady-state vectors \( \bar{A}_e, \bar{V}_e = \bar{\rho}_e/\bar{A}_e^\top \), and \( \bar{u}_e = \bar{U}_e/\bar{A}_e^\top \) and their derivatives \( \bar{V}_{ex}, \bar{u}_{ex} \) using central differences.

Next, we discretize (4.2) and convert it into a matrix linear algebra problem using finite differences. We discretize \( \phi(x_j) \sim \phi_j \), \( \psi(x_j) \sim \psi_j \), and \( \eta(x_j) \sim \eta_j \), where \( x_j = j\Delta x \) for \( j = 1, ..., N \) and \( \Delta x = 2L/(N-1) \). In
this way, we obtain the matrix eigenvalue problem
\[ \lambda \vec{\xi} = A^{-1}(M + F)\vec{\xi}, \]
\[ (4.15) \]
where \( \vec{\xi} \equiv \left( \begin{array}{c} \vec{\phi} \\ \vec{\psi} \\ \vec{\eta} \end{array} \right) \) is a 3N dimensional vector with \( \vec{\phi} \equiv \left( \begin{array}{c} \phi_1 \\ \vdots \\ \phi_N \end{array} \right), \vec{\psi} \equiv \left( \begin{array}{c} \psi_1 \\ \vdots \\ \psi_N \end{array} \right) \) and \( \vec{\eta} \equiv \left( \begin{array}{c} \eta_1 \\ \vdots \\ \eta_N \end{array} \right) \). The three block matrices \( A, F, \) and \( M \) in \[ (4.15) \] are given symbolically by
\[
A = \begin{pmatrix}
I_N & 2\text{Diag}(AeV_e) & 0I_N \\
2\text{Diag}(A^2eV_e) & \text{Diag}(A^2) & 0I_N \\
\tau_q\text{Diag}(A^2_1^{-1}u_e) & 0I_N & \tau\text{Diag}(A^2) \\
\end{pmatrix},
\]
\[
F = \begin{pmatrix}
\text{Diag}(3A^2_1V_e - 1) & \text{Diag}(A^3) & 0I_N \\
\text{Diag}(-3A^2_1V_e - (2 + q)A^1e_1 + qA^2_1u_eV_e) & \text{Diag}(-A^3_1 - u_eA^{q+2}_1) & \text{Diag}(-u_eA^{q+2}) \\
0I_N & 0I_N & 0I_N \\
\end{pmatrix},
\]
\[
M = \begin{pmatrix}
\varepsilon^2\text{Lap} & 0I_N & 0I_N \\
M_{21} & M_{22} & 0I_N \\
M_{31} & 0I_N & M_{33} \\
\end{pmatrix},
\]
where \( I_N \) is the \( N \times N \) identity matrix and where the blocks of \( M \) are defined by
\[
M_{21} = \text{Diag}(2DA_{ex}Vex + 2DA_{ex}Vex) + \text{Diag}(2DA_{ex}Vex)\text{Der},
\]
\[
M_{22} = \text{Diag}(DA^2_{ex})\text{Lap} + \text{Diag}(2DA_{ex}Vex)\text{Der},
\]
\[
M_{31} = \text{Diag}(q(q - 1)A^{q-2}exuex + qA^{q-1}utexex) + \text{Diag}(qDA^{q-1}utexex)\text{Der},
\]
\[
M_{33} = \text{Diag}(DA^q_{ex})\text{Lap} + \text{Diag}(qA^{q-1}utexex)\text{Der}.
\]
Here \( \text{Diag}(v) \) is defined as a square diagonal matrix with the elements of vector \( v \) on the main diagonal, while \( \text{Lap} \) and \( \text{Der} \) denote Laplacian and derivative operators.

For simplicity, and to eliminate the effect of slow spike dynamics, our numerical computations are based on an initial steady-state that contains two boundary spikes, such as shown in the left panel of Figure 10. For this specific pattern, we first use a bisection method to compute the critical value of \( D \) at which the real part of the principal eigenvalue of \[ (4.15) \] crosses through zero. Hopf bifurcation thresholds are also computed for this pattern. In the right panel of Figure 10, we plot the corresponding linear stability phase diagram for the two-boundary spike steady-states in the \( \tau \) versus \( D \) parameter plane. The dot-dashed vertical line in this figure is the hotspot nucleation threshold \( D_{\text{crit},e} \) at which the principal eigenvalue \( \lambda \) of the discrete approximation \[ (4.15) \] crosses through zero. For \( D > D_{\text{crit},e} \) (region D of Figure 10), the two-boundary spike steady-state is unstable owing to a positive real eigenvalue of \[ (4.15) \]. This initial instability, termed a competition instability, has the effect of triggering a nonlinear event that monotonically annihilates one of the two boundary spikes. In the unshaded region \( C \) of Figure 10 where \( D < D_{\text{crit}} \) and \( \tau \) exceeds a threshold, the two-boundary spike steady-state is unstable to an asynchronous oscillatory instability in the spike amplitudes. This initial instability triggers a nonlinear event leading to the oscillatory collapse of one of the two boundary spikes, with the final pattern consisting of only one remaining boundary spike. This single-boundary spike is linearly stable and persists for all subsequent time. In Figure 11 we show results from full PDE simulations of \[ (1.5) \] at the marked points in the unshaded regions \( C \) and \( D \), which confirm these predictions from the linear stability phase diagram.

The solid vertical line in the right panel in Figure 10 is the hotspot nucleation threshold \( D_{\text{crit},e} \), given in \[ (2.29) \], for a single-boundary spike pattern. For \( D < D_{\text{crit},e} \), there is no steady-state one-boundary spike solution and we predict that a nucleation event will occur that has the effect of creating a new boundary spike at the other
endpoint of the domain. The dashed curve in the shaded region corresponds to the case where the principal eigenvalue crosses the imaginary axis and it separates the shaded region in Figure 10 into two parts. In the shaded region B, a two-boundary spike steady-state solution is linearly stable to asynchronous oscillations in the spike amplitudes and there is no competition instability. This region corresponds to a sufficiently large police diffusivity. In contrast, in the shaded region A of the phase diagram, an asynchronous oscillatory instability in the amplitudes of the two boundary spikes will occur initially. However, as distinct from the behavior in region C, the resulting large-scale asynchronous oscillations in the spike amplitudes do not lead to a spike annihilation event in which only one spike remains. Instead, whenever a boundary spike has nearly been annihilated, a new spike will be nucleated at this boundary. The overall effect of this nucleation-annihilation behavior is to provide persistent large-scale asynchronous oscillations in the spike amplitudes in which the amplitudes of the two boundary spikes oscillate out of phase. This results in a time-periodic displacement of the intensity of crime hotspots between two adjacent spatial locations, reminiscent to that observed in certain field studies [27]. Since regions A and C occur when $\tau$ is above a threshold, and when the criminal diffusivity $D$ is below a threshold, we conclude that large-scale oscillations in the hotspot amplitudes will occur whenever the police diffusivity $D_p = D/\tau$ is too small. In Figure 12 we show the full PDE simulations of (1.5) at the three marked points shown in the shaded regions A and B of Figure 10, which confirm these predictions.

Figure 10: Left: The steady-state two-boundary spike solution for $D = 1.7, L = 0.5, \alpha = 1, \gamma = 2, U_0 = 1, q = 3$. Right: The linear stability phase diagram in the $\tau$ versus $D$ plane for a two-boundary spike steady-state when $\varepsilon = 0.03, q = 3, L = 0.5, \alpha = 1, \gamma = 2, U_0 = 1$. The dot-dashed vertical line is the competition instability threshold $D_c$, corresponding to a zero eigenvalue crossing, as computed numerically from (4.15) by discretizing the eigenvalue problem (4.2). The solid vertical line is the spike insertion critical value $D_{crit}$ for a one-boundary spike steady-state. A two-boundary spike steady-state solution is unstable in the unshaded regions C, D and is linearly stable in the shaded region B. In the shaded region A, the two-boundary spike pattern exhibits nucleation-annihilation behavior owing to the repeated effects of a near oscillatory collapse of one of the two boundary spikes that is prevented by a hotspot nucleation event that occurs near the boundary. The combination of these two processes leads to very large amplitude spike oscillations that persist for all time.
Figure 11: The hotspot amplitudes computed numerically from (1.5) for a two-boundary spike pattern with $D$ and $\tau$ as indicated at two marked points in the unshaded region in Figure 10. The left panel, for the point in region D of Figure 10 shows a competition instability leading to spike annihilation. The right panel, for a point in region C of Figure 10 shows an asynchronous oscillatory instability in the spike amplitudes that leads to the annihilation of one of the two boundary spikes and a final one-boundary spike steady-state. Parameters: $q = 3$, $L = 0.5$, $\alpha = 1$, $\gamma = 2$, $U_0 = 1$.

Figure 12: The hotspot amplitudes computed numerically from (1.5) for a two-boundary hotspot pattern with $D$ and $\tau$ as indicated in regions A and B of the phase diagram of Figure 10. Left and middle: For Region B, damped asynchronous oscillations in the spike amplitudes lead to a two-boundary spike steady-state. Right: For Region A, large-scale persistent oscillations in the amplitudes of the two-boundary spikes arise owing to the combined effect of an asynchronous oscillatory instability and spike nucleation behavior. Parameters: $q = 3$, $L = 0.5$, $\alpha = 1$, $\gamma = 2$, $U_0 = 1$. 
5 Discussion

We have developed a hybrid asymptotic-numerical approach to study the existence and linear stability of steady-state hotspot patterns for the three-component RD model (1.1) of urban crime, which incorporates the effect of police deployment, in the regime where $D = O(1)$. From a detailed study of a specific two-hotspot pattern, we have identified two opposing qualitative mechanisms, one of hotspot nucleation and the other of hotspot annihilation arising from an apparent subcritical asynchronous oscillation in the hotspot amplitudes. These mechanisms are likely responsible for the occurrence of large-scale irregular temporal oscillations in the hotspot amplitudes that can exist for (1.1) in certain parameter regimes. We emphasize that such intricate spatio-temporal hotspot dynamics are not due to stochastic effects, such as those that are inherent in agent-based modeling [6], but instead result from the two competing deterministic mechanisms of hotspot creation and annihilation. In the overlapping parameter regime where both mechanisms occur, we suggest that the problem of predicting where and when hotspots will appear and then temporarily disappear under the effect of a constant police deployment is largely intractable.

More specifically, in our analysis of the three-component model (1.1) for the regime where $D = O(1)$, we have shown that the branch of one-hotspot steady-state solutions has a saddle-node bifurcation point in $D$ below which there is no one-hotspot steady-state. Near this critical value, we have shown from a normal form analysis that new hotspots are nucleated from a quiescent background. For a multispike pattern the region where nucleation occurs is at the spatial midpoint between two adjacent hotspots when $D$ decreases below a threshold that depends on the level of the overall police deployment $U_0$ and the other parameters in (1.1). We have provided a hybrid asymptotic-numerical approach to obtain an accurate prediction of this saddle-node threshold even when $\varepsilon$ is only moderately small, and we have confirmed our results from the bifurcation software pde2path [24, 26]. This analysis is based largely on the new exact solution in Lemma 1 that yields a highly accurate determination of the hotspot profile. The existence of this saddle-node point is the characteristic feature that allows for the creation of new hotspots from a quiescent background. This analysis extends, and improves, a related analysis in [14] for the two-component RD system with no police in the $D = O(1)$ regime.

With regard to the linear stability of hotspot patterns, we derived an NLEP for a one-hotspot steady-state. From numerical computations of the spectrum of this NLEP we have shown that a one-hotspot steady-state is always linearly stable. This linear stability result for one-hotspot steady-states is qualitatively similar to that found in [13] in the absence of police, as well as in [12] for (1.1) in the regime $D = O(\varepsilon^{-2})$. However, since it is analytically intractable to provide a similar NLEP analysis for multispike patterns, owing to the fact that the steady-state solution is spatially non-uniform between adjacent hotspots, a full numerical approach is needed to obtain a phase diagram in the $\tau$ versus $D$ plane that encapsulates the linear stability results. For the specific, but yet highly illustrative, two-boundary spike pattern our numerically-computed phase diagram showed that distinctly different solution behavior will occur in different ranges of $\tau$ and $D$. For large values of $D$, the hotspot steady-state is unstable to an initial competition instability that triggers a monotonic hotspot annihilation event, without any amplitude oscillations. In an intermediate range of $D$, and when the police diffusivity $D_p = D/\tau$ is too small, an asynchronous oscillatory instability in the hotspot amplitudes will occur owing to a Hopf bifurcation. Our numerical evidence suggests that this Hopf bifurcation is subcritical and will trigger the oscillatory collapse of one of the two boundary spikes. Moreover, for smaller values of $D$, we have shown that in the $\tau$ versus $D$ phase diagram where the two-boundary spike steady-state is unstable to an asynchronous oscillatory instability in the spike amplitudes, but where a one-boundary spike solution does not exist, large-scale asynchronous temporal oscillations in the hotspot amplitudes will occur that will persist in time.

Qualitatively extrapolating our linear stability and nucleation-threshold results to more complex spatial patterns than our simple two-boundary spike pattern, we conjecture that nucleation-annihilation dynamics, leading to large amplitude oscillations of hotspot amplitudes, will occur whenever $\tau$ and $D$ are chosen so that an initial $N$-hotspot steady-state is unstable to an asynchronous oscillatory instability in the hotspot amplitudes, but where a steady-state pattern with fewer hotspots does not exist for that value of $D$. This should occur when the
criminal diffusivity is below both the competition threshold and a saddle-node point, but for a sufficiently sluggish police deployment where the police diffusivity $D_p = D/\tau$ is also below a threshold. In such a parameter regime, whenever a hotspot is near collapse we predict that a new hotspot is nucleated that prevents its annihilation. We conjecture that these competing processes lead to large-scale irregular oscillations in the hotspot amplitudes, whereby hotspots undergo repeated near-annihilation and nucleation events from an otherwise quiescent background. Overall, such highly intricate dynamics for (1.1) provide a rather intractable challenge for predicting when and where hotspots of crime will appear. In Figure 13 (see the movie in the Appendix) we show a full numerical simulation of this more complex nucleation-annihilation behavior for a specific parameter set for an initial pattern of three interior spikes. For this pattern we observe that boundary spikes are intermittently nucleated on the domain boundary, while the interior spikes undergo repeated near oscillatory collapse and nucleation events.

Figure 13: The left two panels are color plots of $A$ and $\rho$, respectively, as time increases. The right panel is the amplitude of $A$ versus time for the middle two hotspots centered at $x = 1.5$ and $x = 3$. Parameters: $\tau = 30$, $D = 2.8$, $\varepsilon = 0.05$, $L = 3$, $q = 3$, $\gamma = 2$, $\alpha = 1$, and $U_0 = 2$ (see Appendix C for the movie).

There are a few specific problems that warrant further investigation. Although our analysis of nucleation behavior still applies to multispike steady-state solutions, and the asymptotic results can be implemented for various ranges of parameters, the linear stability results and phase diagram must be recomputed for each specific hotspot steady-state pattern. As a result, it is rather challenging to identify specific parameter ranges in the $\tau$, $D$, $q$, and $U_0$ parameter space where an initial hotspot steady-state pattern is expected to undergo complex spatio-temporal dynamics owing to the combined effect of hotspot nucleation and hotspot annihilation through oscillatory collapse. Moreover, in our proposed mechanism it is essential to numerically establish that the Hopf bifurcation of the hotspot steady-state is in fact subcritical. In this context, it would be clearly worthwhile to numerically construct phase diagrams in parameter space where highly irregular hotspot dynamics should occur for an initial condition that is near a specific hotspot steady-state pattern.

Finally, it would be interesting to develop a similar hybrid approach to determine whether nucleation-annihilation dynamics are possible in a 2-D spatial domain. Two key challenges in a 2-D setting are that there is no explicit formula for the hotspot profile analogous to that in Lemma 1, and that the determination of the hotspot nucleation threshold cannot be reduced to a quadrature as in the 1-D case.
A Appendix A: The Homoclinic Profile

In this appendix we derive the explicit result in (2.3) of Lemma 1 for the profile of the homoclinic solution to (2.7), which satisfies \( w_{yy} - (w - w_\infty) + (w^3 - w_\infty^3) = 0 \). The first integral of this ODE yields

\[
(w_y)^2 = -2F(w), \quad \text{where} \quad F(w) \equiv \int_{w_\infty}^{w} \left( (s - w_\infty) - (s^3 - w_\infty^3) \right) ds. \tag{A.1}
\]

By integrating \( F(w) \) and factoring the resulting expression, we obtain for \( y < 0 \), where \( w_y > 0 \), that

\[
\sqrt{2}w_y = (w - w_\infty)(w_m - w)^{\frac{1}{2}}(w - w_{ms})^{\frac{1}{2}}, \tag{A.2}
\]

where we have defined

\[
w_m \equiv -w_\infty + \sqrt{2 - 2w_\infty^2}, \quad w_{ms} \equiv -w_\infty - \sqrt{2 - 2w_\infty^2}, \quad w_m + w_{ms} = -2w_\infty, \quad w_mw_{ms} = 3w_\infty^2 - 2. \tag{A.3}
\]

Here \( w_\infty \) is the smallest positive root of \( w^3 - w + b = 0 \), which satisfies \( 0 < w_\infty < 1/\sqrt{3} \) for any \( b \) in \( 0 \leq b < 2/(3\sqrt{3}) \). As a result we have \( w_{ms} < w_\infty < w_m \). We set \( y = 0 \) to be where \( w \) has a maximum, i.e. \( w(0) = w_m \).

By separating variables in (A.2), we can write

\[
\int \frac{dw}{(w - w_\infty)\sqrt{-(w + w_\infty)^2 + 2(2 - 2w_\infty^2)}} = \int \frac{1}{\sqrt{2}} \, dy. \tag{A.4}
\]

We then introduce a new variable \( s \) defined by the positive root of

\[
s^2 = -(w + w_\infty)^2 + (2 - 2w_\infty^2) \quad \text{so that} \quad w + w_\infty = \sqrt{2 - 2w_\infty^2 - s^2}, \tag{A.5}
\]

on the range \( w_\infty \leq w \leq w_m \). On this range of \( w \), we observe that \( s \) is monotone decreasing, and that \( s = \sqrt{2 - 6w_\infty^2} \) when \( w = w_\infty \) and \( s = 0 \) when \( w = w_m \). In terms of \( s \), the left hand side of (A.4) can be written as

\[
LHS = \int \frac{ds}{(\sqrt{2 - 2w_\infty^2 - s^2}) \left( 2w_\infty - \sqrt{2 - 2w_\infty^2 - s^2} \right)} = \frac{1}{2w_\infty} (I_1 + I_2), \tag{A.6}
\]

where \( I_1 \equiv \int \frac{1}{\sqrt{2 - 2w_\infty^2 - s^2}} \, ds \) and \( I_2 \equiv \int \frac{1}{2w_\infty - \sqrt{2 - 2w_\infty^2 - s^2}} \, ds \).

The integral \( I_1 \) is readily calculated as

\[
I_1 = \sin^{-1} \left( \frac{s}{\sqrt{2 - 2w_\infty^2}} \right). \tag{A.7}
\]

To calculate \( I_2 \), we multiply by the conjugate of the integrand to rewrite \( I_2 \) as

\[
I_2 = -\int \frac{\sqrt{2 - 2w_\infty^2 - s^2} + 2w_\infty}{2 - s^2 - 6w_\infty^2} \, ds = I_{21} + I_{22}; \quad I_{21} \equiv \int \frac{2w_\infty}{s^2 + 6w_\infty^2 - 2} \, ds, \quad I_{22} \equiv \int \frac{\sqrt{2 - 2w_\infty^2 - s^2}}{s^2 + 6w_\infty^2 - 2} \, ds. \tag{A.8}
\]

By decomposing \( I_{21} \) into partial fractions, we calculate

\[
I_{21} = -\frac{2w_\infty}{\sqrt{2 - 6w_\infty^2}} \tan^{-1} \left( \frac{s}{\sqrt{2 - 6w_\infty^2}} \right). \tag{A.9}
\]
To evaluate $I_{22}$, we use the following anti-derivative valid for $A > 0$ and $B > 0$:

$$
\int \frac{\sqrt{A - x^2}}{x^2 - B^2} \, dx = -\frac{\sqrt{A - B^2}}{B} \tanh^{-1} \left( \frac{x\sqrt{A - B^2}}{B\sqrt{A - x^2}} \right) - \sin^{-1} \left( \frac{x}{\sqrt{A}} \right).
$$

Identifying $A = 2 - 2w_\infty^2$, $B = \sqrt{2 - 6w_\infty^2}$, and $\sqrt{A - B^2} = 2w_\infty$ we determine $I_{22}$ as

$$
I_{22} = -\frac{2w_\infty}{\sqrt{2 - 6w_\infty^2}} \tanh^{-1} \left( \frac{s}{\sqrt{2 - 6w_\infty^2}} \right) - \tanh^{-1} \left( \frac{s}{\sqrt{2 - 2w_\infty^2}} \right).
$$

(A.10)

Next, we substitute (A.9) and (A.10) into (A.8) to determine $I_2$. Upon using this expression for $I_2$, together with (A.7) for $I_1$, we obtain from (A.6) and (A.4) that

$$
\tanh^{-1} \left( \frac{s}{\sqrt{2 - 6w_\infty^2}} \frac{2w_\infty}{\sqrt{2 - 2w_\infty^2} - s^2} \right) + \tanh^{-1} \left( \frac{s}{\sqrt{2 - 2w_\infty^2}} \right) = -y \sqrt{1 - 3w_\infty^2}.
$$

(A.11)

When $s = 0$, which corresponds to the maximum value $w = w_m$ of the homoclinic, we observe from (A.11) that $y = 0$ as required.

Next, by using the identity

$$
\tanh^{-1}(x) = -\frac{1}{2} \ln \left( \frac{1 - x}{1 + x} \right), \quad \text{for } |x| < 1,
$$

together with (A.5), we can exponentiate (A.11) to obtain

$$
\left( \frac{2 - 6w_\infty^2}{\sqrt{2 - 6w_\infty^2}} \frac{2w_\infty}{2w_\infty s} \right) \left( \frac{\sqrt{2 - 6w_\infty^2} - s}{\sqrt{2 - 6w_\infty^2} + s} \right) = e^{2\kappa y}, \quad \text{where } \kappa \equiv \sqrt{1 - 3w_\infty^2}.
$$

(A.12)

This holds on the range $0 \leq s \leq \sqrt{2 - 6w_\infty^2}$, for which $w_\infty \leq w \leq w_m$.

Finally, we must eliminate $s$ between (A.12) and (A.5) so as to determine $w = w(y)$. To do so, it is convenient to define $\eta$ in $0 \leq \eta \leq 1$ by $s = \eta \sqrt{2 - 6w_\infty^2}$. From (A.12) and (A.5) we obtain the two equations

$$
\left( \frac{\beta - \eta}{\beta + \eta} \right) \left( \frac{1 - \eta}{1 + \eta} \right) = e^{2\kappa y}, \quad \eta^2 (2 - 6w_\infty^2) = -4w_\infty^2 \beta^2 + 2(1 - w_\infty^2),
$$

(A.13)

where we have defined $\beta \equiv (w + w_\infty)/(2w_\infty)$. The first equation in (A.13) can be written as the quadratic

$$
\eta^2 + \beta = \frac{-\eta(\beta + 1)}{\tanh(\kappa y)}.
$$

(A.14)

Upon solving the second equation in (A.13) for $\eta^2$, we substitute into (A.14) to determine $\eta$ as

$$
-\frac{\eta(\beta + 1)}{\tanh(\kappa y)} = \frac{2 - 2w_\infty^2 - 4w_\infty^2 \beta^2 + \beta (2 - 6w_\infty^2)}{2 - 6w_\infty^2} = \frac{2(1 + \beta) - 2w_\infty^2 (2\beta^2 + 3\beta + 1)}{2 - 6w_\infty^2},
$$

$$
= \frac{(1 + \beta)}{2 - 6w_\infty^2} \left( 2 - 2w_\infty^2 (2\beta + 1) \right).
$$

After cancelling the common factor $(\beta + 1)$, we get

$$
\eta = -\tanh(\kappa y) \left( \frac{2 - 2w_\infty^2 - 4w_\infty^2 \beta}{2 - 6w_\infty^2} \right).
$$
Next, we substitute this expression into the second equation of (A.13) to eliminate \( \eta \). Upon recalling that 
\[
\beta \equiv (w + w_\infty) / (2w_\infty),
\]
we get
\[
\tanh^2(\kappa y) \left( 2 - 2w_\infty^2 - 2w_\infty(w + w_\infty) \right)^2 = (2 - 6w_\infty^2) \left( -(w + w_\infty)^2 + 2 - 2w_\infty^2 \right).
\]
By using \( \tanh^2(z) = 1 - \text{sech}^2(z) \), we obtain after some algebra that
\[
\text{sech}^2(\kappa y) \left( 2 - 2w_\infty^2 - 2w_\infty(w + w_\infty) \right)^2 = (2 - 2w_\infty^2 - 2w_\infty(w + w_\infty))^2 - (2 - 6w_\infty^2) \left( -(w + w_\infty)^2 + 2 - 2w_\infty^2 \right),
\]
\[
= (2 - 2w_\infty^2) ((w + w_\infty) - 2w_\infty^2).
\]
Finally, we define \( c(y) \) by
\[
c(y) = \sqrt{\frac{2}{1 - w_\infty^2}} \text{sech}(\kappa y), \quad \text{where} \quad \kappa \equiv \sqrt{1 - 3w_\infty^2},
\]
and we take the positive square root of (A.15b) to get
\[
c \left( 2 - 2w_\infty^2 - 2w_\infty(w + w_\infty) \right) = 2 ((w + w_\infty) - 2w_\infty).
\]
Then, by solving for \( w + w_\infty \), we obtain
\[
w + w_\infty = \frac{c(1 - w_\infty^2) + 2w_\infty}{1 + cw_\infty},
\]
which is equivalent to the result (2.8) in Lemma 1.

### B Appendix B: Calculation of Integral \( J_p \)

In this section, we compute the integrals \( J_p \equiv \int_0^\infty (w^p - w_\infty^p) \, dy \), without having to use the explicit profile for \( w = w(y) \). In our steady-state construction, leading to (2.46), we need to compute \( J_2, J_3, J_4, \) and \( J_5 \). Since (2.7) yields \( w_{yy} - (w - w_\infty) + (w^3 - w_\infty^3) = 0 \) we identify by integrating over \( 0 < y < \infty \) that \( J_1 = J_3 \). Moreover, since \( y > 0 \), we obtain that \( w_y = -[2F(w)]^{1/2} \), where \( F(w) \) was given in (A.1).

#### B.1 Calculation of \( J_1 = J_3 \)

We first write \( J_1 \) as
\[
J_1 \equiv \int_0^\infty (w - w_\infty) \, dy = \int_{w_m}^{w_\infty} (w - w_\infty) \frac{dy}{dw} \, dw = -\int_{w_m}^{w_\infty} \frac{(w - w_\infty)}{\sqrt{-2F(w)}} \, dw = -\sqrt{2} \int_{w_m}^{w_\infty} \frac{dw}{\sqrt{(w_m - w)(w - w_{ms})}}.
\]
By introducing the new variable \( s \) by \( w_m - w = s^2 \), and using (A.3) to calculate \( w_m - w_{ms} \), we get
\[
J_1 = J_3 = 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{ds}{\sqrt{w_m - w_{ms} - s^2}} = 2\sqrt{2} \arcsin \left( \frac{\sqrt{w_m - w_\infty}}{2^{3/4} (1 - w_\infty^2)^{1/4}} \right).
\]
B.2 Calculation of $J_2$

To evaluate $J_2$, we proceed in a similar way to obtain

$$J_2 \equiv \int_0^\infty (w^2 - w_\infty^2) \, dy = - \int_{w_m}^{w_\infty} \frac{(w^2 - w_\infty^2)}{\sqrt{-2F(w)}} \, dw = -\sqrt{2} \int_{w_m}^{w_\infty} \frac{(w + w_\infty)}{\sqrt{(w_m - w)(w - w_m)}} \, dw.$$ 

Then, we introduce $s$ by $w_m - w = s^2$ to obtain

$$J_2 = 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{w_m - w_m - s^2}{\sqrt{w_m - w_m - s^2}} \, ds + 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{w_m + w_\infty}{\sqrt{w_m - w_m - s^2}} \, ds.$$ 

This yields that

$$J_2 = 2\sqrt{2} (w_m s + w_\infty) \sin^{-1} \left( \frac{\sqrt{w_m - w_\infty}}{\sqrt{w_m - w_m}} \right) + 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{w_m - w_\infty}{\sqrt{w_m - w_m - s^2}} \, ds. \quad (B.2)$$

To evaluate the integral in (B.2), we use

$$\int_0^A \frac{\sqrt{B^2 - x^2}}{x} \, dx = \frac{B^2}{2} \sin^{-1} \left( \frac{A}{B} \right) + \frac{A\sqrt{B^2 - A^2}}{2},$$

with $A = \sqrt{w_m - w_\infty}$ and $B = \sqrt{w_m - w_m}$. Then, (B.2) becomes

$$J_2 = \sqrt{2} (w_m s + w_m + 2w_\infty) \sin^{-1} \left( \frac{\sqrt{w_m - w_\infty}}{\sqrt{w_m - w_m}} \right) + \sqrt{2}\sqrt{-w_\infty^2 + w_\infty(w_m + w_m) - w_m w_m}. \quad (B.3)$$

Finally, we use (A.3) to notice that $w_m s + w_m + 2w_\infty = 0$ and to evaluate $w_m w_m$. This yields the simple result

$$J_2 = \int_0^\infty (w^2 - w_\infty^2) \, dy = \sqrt{2}\sqrt{2 - 6w_\infty^2}. \quad (B.3)$$

B.3 Calculation of $J_4$ and $J_5$

The integral $J_4$ is calculated in a similar way as

$$J_4 \equiv \int_0^\infty (w^4 - w_\infty^4) \, dy = - \int_{w_m}^{w_\infty} \frac{(w^4 - w_\infty^4)}{\sqrt{-2F(w)}} \, dw = -\sqrt{2} \int_{w_m}^{w_\infty} \frac{(w^4 - w_\infty^4)/(w - w_\infty)}{\sqrt{(w_m - w)(w - w_m)}} \, dw$$

$$= -\sqrt{2} \int_{w_m}^{w_\infty} \frac{(w^2 + w_\infty^2)(w + w_\infty)}{\sqrt{(w_m - w)(w - w_m)}} \, dw. \quad (B.4)$$

Then, using the substitution $w_m - w = s^2$, $J_4$ becomes

$$J_4 = 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{(w_m + w_\infty - s^2)(w_m - s^2)^2 + w_\infty^2}{\sqrt{w_m - w_m - s^2}} \, ds. \quad (B.5)$$

Similarly, we can rewrite $J_5 \equiv \int_0^\infty (w^5 - w_\infty^5) \, dy$, and by introducing the new variable $w_m - w = s^2$ we obtain

$$J_5 = 2\sqrt{2} \int_0^{\sqrt{w_m - w_\infty}} \frac{(w_m - s^2)^4 + w_\infty(w_m - s^2)^3 + w_\infty^2(w_m - s^2)^2 + w_\infty^3(w_m - s^2) + w_\infty^4}{\sqrt{w_m - w_m - s^2}} \, ds. \quad (B.6)$$

Although there are no explicit expressions for $J_4$ and $J_5$, these two integrals are nonsingular and can be readily calculated numerically.
Appendix C: Illustration by Movies

C.1 Figure 2 (Left): Hotspot Nucleation.

C.2 Figure 2 (Right): Hotspot Annihilation from a Competition Instability.
C.3 
*Figure 3:* Asynchronous Amplitude Oscillations and an Oscillatory Collapse.

[Diagram: Graph showing oscillations with labels for steps and values.]

C.4 
*Figure 13:* Complex Nucleation-Annihilation Dynamics.

[Diagram: Graph showing nucleation and annihilation dynamics with labels for steps and values.]

**Acknowledgements**

We would like to thank Ruicheng Zhao for proposing the idea to generate simulation videos via screen recording, which significantly simplified the process of embedding videos into the PDF format.

**References**


