# Math 110 (Section 002) Learning Objectives

*Students should be able to do the following things by the end of each respective unit. Additions may be made as the term progresses.*

## Lines and Distances

1. Write down the equation of a line in the form $y - y_0 = m(x - x_0)$ given two points, a point and the slope or a graph.

2. Sketch a line given its equation in any form.

3. Determine if lines are perpendicular or parallel.

4. Explain, using a picture, the Pythagorean Theorem.

5. Find the distances between two points on the plane using the Pythagorean Theorem.

6. Write down the equation of a circle in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$ given its radius and centre.

7. Sketch a circle given its equation in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$.

## Functions
*This unit will be expanded throughout the term as your library of functions increases.*

1. Explain what a function is.

2. Give examples of functions and their domains, including functions with restricted or unusual domains (eg. $\mathbb{R} \setminus \{1, 2, 3\}$) or functions whose domains are sets other than real numbers.

3. Calculate the domain of functions of the form $\frac{p(x)}{q(x)}$ where $p$ and $q$ are compositions of polynomials and square roots.

4. Sketch the functions $x$, $x^2$, $x^3$, $\sqrt{x}$, $\frac{1}{x}$ and $|x|$.

5. Sketch the graphs of functions obtained from $x$, $x^2$, $x^3$, $\sqrt{x}$, $\frac{1}{x}$ and $|x|$ via one or more of the following transformations: vertical and horizontal shifts, vertical and horizontal dilations and reflections across the vertical and horizontal axes.

6. Explain, using a picture or examples, what the composition of two or more functions means.

7. Explain, using an example, why function composition is not commutative.

## Velocity and the idea of a limit

1. Explain, using and picture but not the formal vocabulary of limits, what velocity and acceleration are.

2. Sketch the graphs of velocity and acceleration of a particle given a graph of its position.

3. Given graphs of the position, velocity and acceleration of a particle, identify which is which.

4. Numerically estimate the velocity of a particle given a graph of its position.

5. Explain, using a picture, what a difference quotient is.

## Evaluating limits

1. Explain, using a picture and rigorous but informal language, what the following phrases mean,

$$\lim_{x \to a} f(x) = L$$
$$\lim_{x \to a^-} f(x) = L$$
$$\lim_{x \to a^+} f(x) = L.$$

2. Argue why the notion of a limit is important.

3. Calculate the limits of simple rational functions whose limits exit.

4. Give examples of rational functions whose limits at a given finite number of points do not exist.

## Continuity

1. Define what it means for a function to be continuous.

2. Identify where a given function is continuous/discontinuous.

3. Determine the parameters of a piecewise function that makes a function continuous.

4. State and explain the Intermediate Value Theorem.

5. Give examples of functions that do not satisfy the Intermediate Value Theorem by virtue of their discontinuity either in the interior of an interval or at its endpoints.

6. Use the Intermediate Value Theorem to estimate roots of functions, including non-polynomial functions.

7. Use the Bisection Algorithm to estimate roots of functions to desired accuracy.

8. Use the Intermediate Value Theorem to construct short proofs (eg. Prove that a continuous function $f : [0, 1] \to [0, 1]$ has a fixed point).

## Velocity and the idea of derivative

1. Identify that the slope of a tangent line is a limit.

2. Estimate slopes of tangent lines using the definition of derivative.

**The definition of derivative**

1. Define the derivative of a given function using limits.

2. Calculate derivatives, using the limit definition, of polynomial functions of degree $d \leq 3$, and of rational functions $\frac{p(x)}{q(x)}$ where $p$ and $q$ are linear.

3. Find the equation of lines tangent to the graphs of polynomial functions of degree $d \leq 3$, and of rational functions $\frac{p(x)}{q(x)}$ where $p$ and $q$ are linear.

4. Define what it means for a function to be differentiable.

5. Explain why differentiable functions are continuous.

6. Demonstrate, using an example, that continuous functions need not be differentiable.

7. Explain why the derivative a function is itself a linear function.

**The Power, Product and Quotient Rules**

1. Explain why the term $(x+h)^n$ may be expanded as $x^n + na^{n-1}h + h^2 f(x, h)$ where $f$ is a polynomial in $x$ and $h$.

2. Show, using the limit definition of derivative and the "pseudo-binomial theorem" above, why the Power Rule holds for integer values of $n$.

3. Explain the steps of the proofs of the Product and Quotient Rules, given the proof.

4. Demonstrate the consistency between the Power, Product and Quotient Rules.

5. Differentiate functions using the Power, Product and Quotient Rules.

6. Find tangent lines, normal lines and horizontal tangent lines to curves using the Power, Product and Quotient Rules.

**Trigonometry**

1. Argue for the "naturalness" of measuring angles in radians as opposed to degrees.

2. Argue for the usefulness of trigonometric functions.

3. Define the functions $\sin x$ and $\cos x$ in terms of the unit circle as well as right triangles.

4. Sketch the graphs of $\sin x$ and $\cos x$ as well as transformations of $\sin x$ and $\cos x$.

5. Explain what the derivatives of $\sin x$ and $\cos x$ are and justify them without proof.

6. Apply derivative rules to functions that are a composition of trig functions and polynomials.

## Exponentials

1. Explain what a differential equation is.

2. Define the number $e$ and sketch the function $e^x$ (possibly subject to some transformations).

3. Define and sketch the function $\ln x$ (again with transformations).

4. Argue for the "naturalness" of base-$e$ as opposed to base-10 logarithms.

5. Do arithmetic with exponentials $\left(eg.\ \text{simplify } \left(2^3\right)^4 2^2\right)$.

6. Argue for the usefulness of exponential and logarithmic functions in terms of modeling and differential equations.

7. Explain what the derivatives of $e^x$ and $\ln x$ are and justify them without proof.

## Chain Rule

1. Explain the steps of the proof of the Chain Rule, given a simplified version of the proof.

2. Explain using examples, why the Chain Rule is plausible.

3. Differentiate functions using the Chain Rule.

4. Find tangent lines, normal lines and horizontal lines to curves using the Chain Rule.

5. Differentiate $a^x$ and $\log_b x$ for constants $a$ and $b$.

## Implicit Differentiation

1. Find $\frac{dy}{dx}$ using implicit differentiation, given a curve $f(x, y) = g(x, y)$ composed of simple functions.

2. Justify the technique of implicit differentiation using chain rule.

3. Use logarithmic differentiation to differentiate products of functions, quotients of functions or functions with variables in both the base and the exponent (such as $x^x$).

## Extrema
*Note that curve sketching will be introduced over the following weeks. The unit' Curve Sketching' is a summary unit.*

1. Define the terms 'critical point', ''local maximum', 'local minimum', 'global maximum' and 'global minimum'.

2. Identify critical points and find global extrema of continuous, almost-everywhere-differentiable functions, including piecewise functions, on closed intervals.

3. State and explain the Extreme Value Theorem.

4. Give examples of functions that do not satisfy the Extreme Value Theorem.

5. Prove that certain simple functions (eg. cubics) have no global extrema on the interval $(-\infty, \infty)$.

## The Mean Value Theorem

1. State and explain the Mean Value Theorem.

2. Define what it means for a function to be increasing or decreasing.

3. Explain how the Mean Value Theorem connects the derivative of a function with the graph of the function.

4. Explain the proof of the Mean Value Theorem, give and outline of the proof or a picture.

5. Apply the Mean Value Theorem to velocity problems, including problems where the theorem must be applied to a difference of functions (eg. Prove that if two runners end a race in a tie then there must have been a point in time where both runners had the same speed).

## The First and Second Derivative Test

1. State and explain the First and Second Derivative Tests.

2. Use the First and Second Derivative Tests to find local and global extrema of differentiable functions (or possibly functions that are not differentiable at a few points).

3. Given a differentiable function, identify its intervals of increase and decrease as well as its extrema (again perhaps with a few points of non-differentiability).

## Concavity

1. Define the terms 'concave up', 'concave down' and 'inflection point'.

2. Explain the connection between the concavity of a function and the sign of its second derivative.

3. Give examples of functions demonstrating the distinction between the qualities concave up/concave/down and increasing/decreasing.

4. Given a function, identify its inflection points and intervals of concavity.

## Asymptotes

1. Explain, using a picture and rigorous but informal language, what the phrase $\lim_{x \to a} f(x) = L$ means if $a = \pm\infty$ or $L = \pm\infty$.

2. Define the terms 'vertical asymptote' and 'horizontal asymptote'.

3. Give examples of functions with vertical or horizontal asymptotes.

4. Given a function, identify its asymptotes.

5. State and apply l'Hospital's Rule on functions already in the appropriate form $\frac{f(x)}{g(x)}$.

6. Give some justification for l'Hospital's Rule.

## Curve Sketching

1. Sketch the graph of a given function, indicating its domain, intercepts, and asymptotes; intervals of increase/decrease and extrema; and intervals of concavity and inflection points.

2. Come up with the graph of, and an algebraic expression for, a (possibly piecewise) function, given any combination of the following: domain, intercepts, asymptotes, intervals of increase/decrease, extrema, intervals of concavity, and inflection points.

3. Define what it means for a function to be 'even' or 'odd'.

4. Identify a given function as even or odd. Give examples of even and odd functions.

## Optimization

1. Explain what a mathematical model is, and give examples of models.

2. Give examples of 'real life' optimization problems in his/her intended field.

3. Solve optimization problems that reduce to analyzing the graph of a function of one variable with a small number of discontinuities, using a systematic method of enumerated steps.

4. Evaluate the suitability of a model based on dimensions, range, and extremal behaviour.

## Related Rates

1. Give examples of "real life" related rates problems in his/her intended field.

2. Solve simple related rates problems using a systematic method of enumerated steps.

3. Evaluate the plausibility of a model involving derivatives based on dimensions, range, and extremal behaviour.

## Approximation

1. Find the linear approximation to a function $f$ at $a$, given functions for which the derivative $f'(a)$ can be found.

2. Identify good candidates for 'a' when using linear approximation to determine $f(x) = f(a + small)$.

3. Sketch the graph of a linear approximation to a function at a point, and use the graph to determine the sign of the remainder.

4. User linear approximation to estimate predictions of nonlinear mathematical models.

5. Explain why a linear approximation is generally better than a constant approximation.

6. Find the approximations of degree 'n' for $e^x$, $\sin x$ and $\cos x$ about $a = 0$.

7. Use Newton's Method to approximate roots of functions.