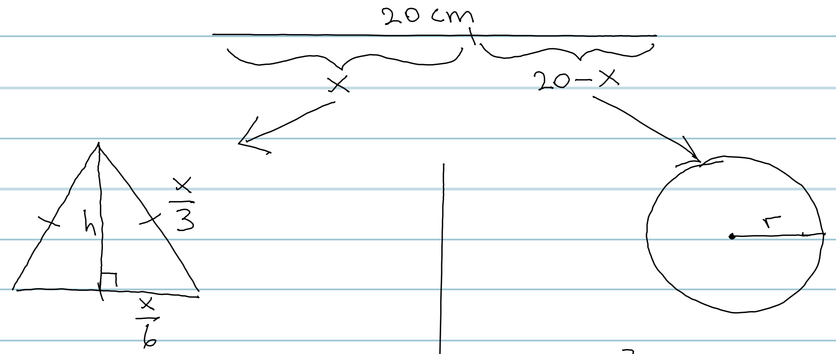
Optimization: Playing with Wire

A length of wire 20cm long is cut into two pieces, one of which is bent into the shape of an equilateral triangle, and the other of which is bent into the shape of a circle. Find the length of wire used in the triangle when the sum of the shapes' areas is a: a) maximum b) minimum.



$$A = \frac{1}{2} \underset{\substack{\uparrow \\ \frac{x}{3}}}{b} \cdot \underset{\substack{\uparrow \\ ?}}{h}$$

$$\text{height: } h^2 + \left(\frac{x}{6}\right)^2 = \left(\frac{x}{3}\right)^2$$

$$h^2 = \left(\frac{x}{3}\right)^2 - \left(\frac{x}{6}\right)^2$$

$$h = \sqrt{\left(\frac{x}{3}\right)^2 - \left(\frac{x}{6}\right)^2}$$

$$= \sqrt{\frac{(4)x^2}{(4)9} - \frac{x^2}{36}}$$

$$= \sqrt{\frac{4x^2}{36} - \frac{x^2}{36}}$$

$$= \sqrt{\frac{3x^2}{36}}$$

$$= \sqrt{\frac{x^2}{12}}$$

$$= \frac{x}{2\sqrt{3}} \frac{(\sqrt{3})}{(\sqrt{3})}$$

$$= \frac{\sqrt{3}x}{6}$$

$$A = \frac{1}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{3}x}{6}$$

$$= \frac{\sqrt{3}x^2}{36}$$

$$A = \pi r^2$$

$$= \pi\left(\frac{20-x}{2\pi}\right)^2$$

$$20 - x = 2\pi r$$

$$r = \frac{20-x}{2\pi}$$

Total Area: $A(x) = \frac{\sqrt{3}x^2}{36} + \pi\left(\frac{20-x}{2\pi}\right)^2$

$$A'(x) = \frac{2\sqrt{3}x}{36} + \cancel{2\pi}\left(\frac{20-x}{\cancel{2\pi}}\right)\frac{d}{dx}\left(\frac{20-x}{2\pi}\right)$$

$$= \frac{\sqrt{3}x}{18} + (20-x)\left(\frac{-1}{2\pi}\right)$$

$$0 = \frac{\sqrt{3}x}{18} + \frac{x-20}{2\pi}$$

$$0 = \frac{\sqrt{3}x}{2\cdot 9}\left(\frac{\pi}{\pi}\right) + \frac{x-20}{2\cdot\pi}\left(\frac{9}{9}\right)$$

$$0 = \frac{\sqrt{3}\pi x}{18\pi} + \frac{9x-180}{18\pi}$$

$$0 = \frac{\sqrt{3}\pi x + 9x - 180}{18\pi}$$

$$0 = \sqrt{3}\pi x + 9x - 180$$

$$180 = \sqrt{3}\pi x + 9x$$

$$180 = x(\sqrt{3}\pi + 9)$$

$$x = \frac{180}{\sqrt{3}\pi + 9} \sim 12{,}46 \text{ cm}$$

$$A(12{,}46) \sim 12 \text{ cm}^2$$

Check Endpoints: $0 \leq x \leq 20$

$$A(0) \sim 31{,}83 \text{ cm}^2$$
$$A(20) \sim 19{,}25 \text{ cm}^2$$

$$\underbrace{A(0)}_{\text{max}} > A(20) > \underbrace{A(12{,}46)}_{\text{min}}$$

a) The length of wire in the triangle to maximize total area is 0 cm.

b) The length of wire in the triangle to minimize total area is $\frac{180}{\sqrt{3}\pi+9}$ cm.