

## ASSIGNMENT 1 · 6 SOLUTIONS, Section 002

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There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.

1. Let  $f(x)$  be the function defined by:

$$f(x) = \begin{cases} 2x + 5 & x < -2 \\ 2ax^2 + b & -2 \leq x < 5 \\ 7 - 3x & x \geq 5 \end{cases}$$

Find the values of  $a$  and  $b$  that make this function continuous.

SOLUTION: Since each subfunction of  $f(x)$  is a polynomial, they are all continuous in their specified intervals. The only place the function may not be continuous are at the  $x$ -values where we switch piecewise functions, i.e. at  $x = -2$  and  $x = 5$ . We need to make sure that the first piece  $2x + 5$  is equal to  $2ax^2 + b$  when  $x = -2$ , as well as making sure that  $7 - 3x$  equals  $2ax^2 + b$  at  $x = 5$ . So, we have:

$$\begin{aligned} 2(-2) + 5 &= 2a(-2)^2 + b \\ 7 - 3(5) &= 2a(5)^2 + b \end{aligned}$$

Simplifying each equation, we have the following system:

$$1 = 8a + b \tag{1}$$

$$-8 = 50a + b \tag{2}$$

We will solve this system of equations by *elimination*. Multiplying the second line (2) by  $-1$  and subtracting, we have:

$$\begin{aligned} 9 &= -42a \\ a &= \frac{-3}{14} \end{aligned}$$

By *substituting* the value  $a = \frac{-3}{14}$  into either (1) or (2), we may obtain the value of  $b$ . So,

$$\begin{aligned} 1 &= 8\left(\frac{-3}{14}\right) + b \\ b &= \frac{19}{7} \end{aligned}$$

Finally, we have that  $a = \frac{-3}{14}$  and  $b = \frac{19}{7}$ . Substituting this back into the original equation in the piece-wise function  $f(x)$ , we have:

$$f(x) = \begin{cases} 2x + 5 & x < -2 \\ \frac{-3}{7}x^2 + \frac{9}{7} & -2 \leq x < 5 \\ 7 - 3x & x \geq 5 \end{cases}$$

Just to be careful, let's check that our solution is correct by plugging  $x = -2$  and  $x = 5$  into  $f_2(x) = \frac{-3}{7}x^2 + \frac{9}{7}$ .

$$\begin{aligned}\frac{-3}{7}(-2)^2 + \frac{9}{7} &= \frac{-12}{7} + \frac{19}{7} = 1 \\ \frac{-3}{7}(5)^2 + \frac{9}{7} &= \frac{-75}{7} + \frac{19}{7} = -8\end{aligned}$$

So, the  $a$  and  $b$  that we found do indeed make  $f(x)$  continuous.

2. Consider the function

$$f(x) = x^3 - 2x^2 - 2x + 2.$$

- (a) Show that  $f(x)$  has at least 3 roots (zeros).
- (b) Pick one of the roots and using the Bisection Algorithm find an interval of size 0.1 or smaller containing the root. Include the results of each iteration (feel free to use a calculator).
- (c) **Bonus** Compute, using the Bisection Algorithm one of the zeros to three decimal places. There will be a prize for the person who computes the most decimal places of a zero (if you use a computer it must be programmed by yourself).

SOLUTION: (a) Our goal is to apply the Intermediate Value Theorem (IVT). We know that  $f$  is a polynomial and hence continuous. We compute,

$$f(-2) = -10, \quad f(0) = 2, \quad f(1) = -1, \quad f(3) = 5.$$

Using the first two computations we conclude via the IVT that there exists an  $x_0$  between  $-2$  and  $0$  such that  $f(x_0) = 0$ . Similar applications of IVT with the remaining computations assert the existence of a root,  $x_1$ , between  $0$  and  $1$  and a root,  $x_2$ , between  $1$  and  $3$ . Hence  $f$  has at least 3 roots, the desired result.

(b) We pick the root between  $0$  and  $1$ ,  $x_1$ . Identical arguments can be used to narrow the location of roots  $x_0$  and  $x_2$ . In what follows we compute the value of  $f$  at the midpoint of the interval achieved in the previous step. Based on the sign of this number and the information contained in the previous step we cut the interval containing the root in half. Note that at each step we apply the IVT.

- Step 0:  $f(0) = 2 > 0$  and  $f(1) = -1 < 0 \Rightarrow x_1 \in (0, 1)$
- Step 1:  $f(0.5) = 0.625 > 0$  (and  $f(1) < 0$ )  $\Rightarrow x_1 \in (0.5, 1)$
- Step 2:  $f(0.75) = -0.203125 < 0$  (and  $f(0.5) > 0$ )  $\Rightarrow x_1 \in (0.5, 0.75)$
- Step 3:  $f(0.625) = \dots > 0$  (and  $f(0.75) < 0$ )  $\Rightarrow x_1 \in (0.625, 0.75)$
- Step 4:  $f(0.6875) > 0 \Rightarrow x_1 \in (0.6875, 0.75)$

At this point we notice that our interval  $(0.6875, 0.75)$  has size  $0.0625$  which is less than  $0.1$ . So, we have found an interval of size smaller than  $0.1$  containing  $x_1$ .

(c) We repeat the arguments of part (b) to narrow the interval further,

- Step 5:  $f(0.71875) < 0 \Rightarrow x_1 \in (0.6875, 0.71875)$   
 Step 6:  $f(0.703125) < 0 \Rightarrow x_1 \in (0.6875, 0.703125)$   
 Step 7:  $f(0.6953125) < 0 \Rightarrow x_1 \in (0.6875, 0.6953125)$   
 Step 8:  $f(0.69140625) < 0 \Rightarrow x_1 \in (0.6875, 0.69140625)$   
 Step 9:  $f(0.689453125) < 0 \Rightarrow x_1 \in (0.6875, 0.689453125)$   
 Step 10:  $f(0.6884765625) > 0 \Rightarrow x_1 \in (0.6884765625, 0.689453125)$   
 Step 11:  $f(0.68896484375) < 0 \Rightarrow x_1 \in (0.6884765625, 0.68896484375)$

Note that you could have rounded at any step to make the numbers nicer. With this last step we see that the first three decimal places of our zero will be 0.688. We can continue to find more decimal places. Using a computer program to repeat this algorithm to a desired level of accuracy we compute each zero to 15 decimal places,

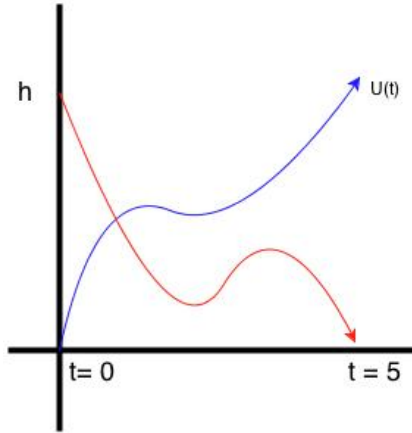
$$\begin{aligned}x_0 &= -1.170086486626034 \\x_1 &= 0.688892182534018 \\x_2 &= 2.481194304092016\end{aligned}$$

3. The other weekend, I decided to go for a hike. At the bottom of the mountain around 10 : 00am, I started my long hike up the mountain. Along the way, I made stops to eat, relax, and even had to go backwards to pick up objects I had dropped earlier. I finally reached the summit at 3 : 00pm. I decided to stay the night at the summit and leave to go back down at 10 : 00am the next day. Taking the same track down with stops and detours at various times, I made it back down to the point I started from the day before at 3 : 00pm.

Prove that there existed a time where I was at the exact same location on the mountain at the exact same time on *both days*. You should provide a precise mathematical argument as well as a picture depicting the situation.

HINT: Consider the difference in height at time  $t$  where  $t$  is the time after 10am.

SOLUTION: The key to this problem is the Intermediate Value Theorem. Let  $U(t)$  be the function (with respect to time,  $t$ ) representing my position as I hike *up*, and let  $D(t)$  be the function representing my position as I go *down* the mountain. Since I'm never teleporting down the mountain, the path I take has to be continuous. So,  $U(t)$  and  $D(t)$  are both continuous functions on my time interval of 5 hours. If I graph  $U(t)$  and  $D(t)$ , it is obvious that no matter how I draw them, they must intersect at some point in time,  $t_0$ . This intersection means that  $(t_0, U(t_0)) = (t_0, D(t_0))$ , i.e. there is a time where I was at the exact same location on the mountain at the exact time on both days! The graph looks like this:



Now, let's solve this with a mathematical argument. We will define  $f(t) = U(t) - D(t)$ . Since  $U(t)$  and  $D(t)$  are both continuous, it is true that their difference  $f(t)$  is continuous as well. If  $h$  is the height of the mountain and if we measure time in hours, we know from the information given that:

$$\begin{aligned} U(0) = 0 \text{ and } U(5) = h \\ \text{and} \\ D(0) = h \text{ and } D(5) = 0 \end{aligned}$$

Hence, for  $f(t)$ , we have that:

$$f(0) = U(0) - D(0) = 0 - h = -h \tag{3}$$

$$f(5) = U(5) - D(5) = h - 0 = h \tag{4}$$

So, by the Intermediate Value Theorem, since  $f(t)$  is continuous on  $[0, 5]$  and  $f(0) = -h \leq 0 \leq h = f(5)$ , we have that there exists some value  $t_0$ ,  $0 \leq t_0 \leq 5$ , with  $f(t_0) = 0$ .

Since  $f(t_0) = u(t_0) - D(t_0) = 0$ , we have that  $U(t_0) = D(t_0)$ , meaning that the functions  $U$  and  $D$  do intersect at some time  $t_0$ .