Instructions:

- Completion of this assignment optional and will not be counted for grades.
- This assignment covers topics that will be on the final exam.
- The assignment can be submitted for "grading" at the start of class on the date stated above.
- Submission of this assignment optional.

Questions:

1. Evaluate the following limits using any method covered in class. Conclude whether the functions have any asymptotes at the following points. *Hint: Consider rearranging the equation first.*

   (a)

$$\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)}.$$

Solution: In order to tackle limits, we try the simply thing first. If we directly substitute, we get:

$$\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} = \frac{2\sin(0) - \sin(0)}{0 - \sin(0)}$$

which gives the indeterminate form "$\frac{0}{0}$". So that means we should try L'Hopital's rule.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &= \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sin(x) - \sin(2x)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(x - \sin(x)\right)} \\ &= \lim_{x\to 0} \frac{2\cos(x) - 2\cos(2x)}{1 - \cos(x)}\end{aligned}$$

which is still an indeterminate form when we evaluate it, so we try L'Hopital's again

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &= \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(2\cos(x) - 2\cos(2x)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \cos(x)\right)} \\ &= \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)}{\sin(x)}\end{aligned}$$

This is looking good as we are now left with simplifying trig functions. We make use of the double angle formula:

$$\sin(2x) = 2\sin(x)\cos(x)$$

to get

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &= \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)}{\sin(x)} \\ &= \lim_{x\to 0} \frac{-2\sin(x) + 4\left(2\sin(x)\cos(x)\right)}{\sin(x)} \\ &= \lim_{x\to 0} \frac{-2\sin(x) + 8\sin(x)\cos(x)}{\sin(x)} \\ &= \lim_{x\to 0} \frac{\sin(x)\left(-2 + 8\cos(x)\right)}{\sin(x)} \\ &= \lim_{x\to 0} -2 + 8\cos(x) = 6\end{aligned}$$

So that means the initial limit must be 6. Further, we can conclude that there are no vertical asymptotes at $x = 0$ since $\lim_{x\to 0} f(x) \neq \pm\infty$.

Solution: For this problem, I had initially intended for students to look for asymptotes at the given points. But for completeness, lets find all the asymptotes. In this case, we see that the numerator does not get any bigger than 2 for any x. But as x gets really big, the denominator gets really large (either positively or negatively). So that means the denominator gets to infinite first. Hence we have:

$$\lim_{x\to\infty} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} = 0 \quad \lim_{x\to-\infty} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} \qquad = 0$$

So we have a horizontal asymptotes in both directions at $y = 0$. Finally, since the denominator is not zero if $x \neq 0$, we don't have any vertical asymptotes anywhere.

(b)

$$\lim_{x\to 0} x \ln(x).$$

Solution: If we try direct substitution, we would get something like "$0 \cdot \infty$", which is just as absurd. So that means we should massage the equation a little bit:

$$\lim_{x\to 0} x \ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}}$$

Aside: This is an semi-arbitrary choice. If we tried this with $\frac{x}{\frac{1}{\ln(x)}}$, we end up getting something more complicated. So this suggests our initial choice might be a better one. Similarly, we can define $u = \frac{1}{x}$ and rewrite the equation. But that means we have to redefine our limit as $u \to \infty$. This method is a little tricky because in theory we need $\pm\infty$. But since the domain here is $(0, \infty)$, we really just need $x \to 0^+$ which translates to $u \to \infty$. Back to the original equation, if we try direct substitution now, we would get "$\frac{0}{0}$", which would scream out L'Hopital's rule. So lets do this again:

$$\begin{aligned}
\lim_{x\to 0} x \ln(x) &= \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}} \\
&= \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x} \ln(x)}{\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x}} \\
&= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0} -x \\
&= 0
\end{aligned}$$

So that means there are no vertical asymptote at $x = 0$.

Solution: For completeness, we see that the domain of this function is $(0, \infty)$. So that means we only have to worry about the horizontal asymptotes as $x \to \infty$. Since,

$$\lim_{x\to 0} x = \infty \qquad \lim_{x\to 0} \ln(x) = \infty.$$

The product of two really big numbers is also a really big number, so we have

$$\lim_{x\to 0} x \ln(x) = \infty$$

implying that there is no horizontal asymptote.

(c)

$$\lim_{x\to -\infty} \sqrt{x^2 - 2x} + x.$$

Solution: Since we are taking a limit to $-\infty$, we apply the one trick we know and switch x to $-x$. For clarity, I will redefine $-u = x$ to get:

$$\begin{aligned}
\lim_{x\to -\infty} \sqrt{x^2 - 2x} + x &= \lim_{u\to\infty} \sqrt{(-u)^2 + 2(-u)} + (-u) \\
&= \lim_{u\to\infty} \sqrt{u^2 - 2u} - u
\end{aligned}$$

If we try substituting ∞, we get an indeterminate "$\infty - \infty$", so we need to massage it a little bit. Start by eliminating the square root:

$$\begin{aligned}
&= \lim_{u\to\infty} \left(\sqrt{u^2 - 2u} - u\right) \cdot \frac{\sqrt{u^2 - 2u} + u}{\sqrt{u^2 - 2u} + u} \\
&= \lim_{u\to\infty} \frac{\left(\sqrt{u^2 - 2u} - u\right)\left(\sqrt{u^2 - 2u} + u\right)}{\sqrt{u^2 - 2u} + u} \\
&= \lim_{u\to\infty} \frac{\left(u^2 - 2u\right) - \left(u^2\right)}{\sqrt{u^2 - 2u} + u} \\
&= \lim_{u\to\infty} \frac{-2u}{\sqrt{u^2 - 2u} + u}
\end{aligned}$$

Now we have converted the original function into a rational one. Lucky for us, it's in indeterminate form as well. So we just hit it with L'Hopital's rule till a number comes out (hopefully).

$$\begin{aligned}
&= \lim_{u\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}u}(-2u)}{\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u^2 - 2u} + u\right)} \\
&= \lim_{u\to\infty} \frac{-2}{\frac{u-1}{\sqrt{u^2-2u}} + 1}
\end{aligned}$$

At this point, we can apply the other trick we know, which is to divide by the highest exponent.

$$= \lim_{u\to\infty} \frac{\frac{-2}{u}}{\frac{\frac{u}{u}-\frac{1}{u}}{\frac{\sqrt{u^2-2u}}{u}} + \frac{1}{u}}$$
$$= \lim_{u\to\infty} \frac{\frac{-2}{u}}{\frac{1-\frac{1}{u}}{\sqrt{1-\frac{2}{u}}} + \frac{1}{u}}$$

At this point, we can look at what happens when u gets really big to get a limit of 0. So that means the original limit must be 0, implying a horizontal asymptote at $y = 0$.

Solution: To check other limits, we should first find the domain of the function. This function breaks when the bit inside the square root is less than zero. So we check:

$$x^2 + 2x = x(x + 2)$$

So that means we want to partition the interval as $(-\infty, 0), (0, 2), (2, \infty)$. Checking each interval, we see that the interval $(-2, 0)$ is the only one that yields negative results. So the domain of the function is $(-\infty, -2] \cup [0, \infty)$ since a zero output is allowed. We can check that the function is continuous on the domain, so that means:

$$\lim_{x\to 2^+} \sqrt{x^2 - 2x} + x = \sqrt{(2)^2 - 2(2)} + (2) \qquad = 2$$
$$\lim_{x\to 0^-} \sqrt{(0)^2 - 2(0)} + (0) = 0$$

So this means there are no vertical asymptotes at the discontinuities of this function (and no vertical asymptotes overall). To check for the other horizontal asymptote, we take $x \to \infty$. Looking at the function, we see that as x gets really big, we are adding two really big numbers together. So that means the overall sum will keep getting bigger implying that there is no horizontal asymptote on the positive infinity side.

2. Hand-sketch the following functions using the steps covered in lectures (Be sure to highlight all important points/trends in your sketch).

   (a)
$$f(x) = \frac{x^2 + 4}{x^2 - 4}.$$

Solution: Time to get this party started. Let just go through all the steps:

1. **Domain:** Since this is a rational function, we check where the denominator is zero. Solving $x^2 - 4 = 0$, we have that $x = \pm 2$. So the domain $D = \mathbb{R} - \{-2, 2\}$

2. **Symmetry:** We simply evaluate:

$$\begin{aligned} f(-x) &= \frac{(-x)^2+4}{(-x)^2-4} \\ &= \frac{x^2+4}{x^2-4} = f(x) \end{aligned}$$

So that means the function is even. Yay, half the work done.

3. y**-intercept:** We simply have $f(0) = \frac{0+4}{0-4} = -1$.

4. x**-intercept:** We want to solve the when $f(x) = 0$. In the case of rational function, this is simply when the numerator is zero. $x^2 + 4 = 0$. We see that there are no solutions for x. So that means, we have no x-intercepts.

5. **Sign:** Since there are no roots, we simply partition by the positive discontinuities

Interval	$f(x)$	POS/NEG
$(0,2)$	$f(1) < 0$	NEG
$(2,\infty)$	$f(4) > 0$	POS

6. **Horizontal Asymptote:** We just have to take a limit to infinity:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2+4}{x^2-4} &= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} \\ &= \lim_{x\to\infty} \frac{1+\frac{4}{x^2}}{1-\frac{4}{x^2}} \\ &= \frac{1}{1} = 1 \end{aligned}$$

So we have a horizontal asymptote at $y = 1$.

7. **Vertical Asymptote:** To check vertical asymptotes, we have to take two limits:

$$\begin{aligned} \lim_{x\to 2^-} \frac{x^2+4}{x^2-4} &= \frac{8}{0^-} = -\infty \\ \lim_{x\to 2^+} \frac{x^2+4}{x^2-4} &= \frac{8}{0^+} = +\infty \end{aligned}$$

So we have a vertical asymptote at $x = 2$.

8. **Intervals of INC/DEC:** To get the intervals of increase and decrease, we need to find the derivative and critical points.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \frac{x^2+4}{x^2-4} &= \frac{2x(x^2-4) - 2x(x^2+4)}{(x^2-4)^2} \\ &= \frac{-16x}{(x^2-4)^2} \end{aligned}$$

So that means the only critical points are $x = 0$. We can then partition the interval by critical points and discontinuities as follows:

Interval	$f'(x)$	INC/DEC
$(0,2)$	$f'(1)<0$	DEC
$(2,\infty)$	$f'(4)<0$	DEC

9. **Intervals of Concavity:** To get the intervals of concavity, we make use of the second derivative:

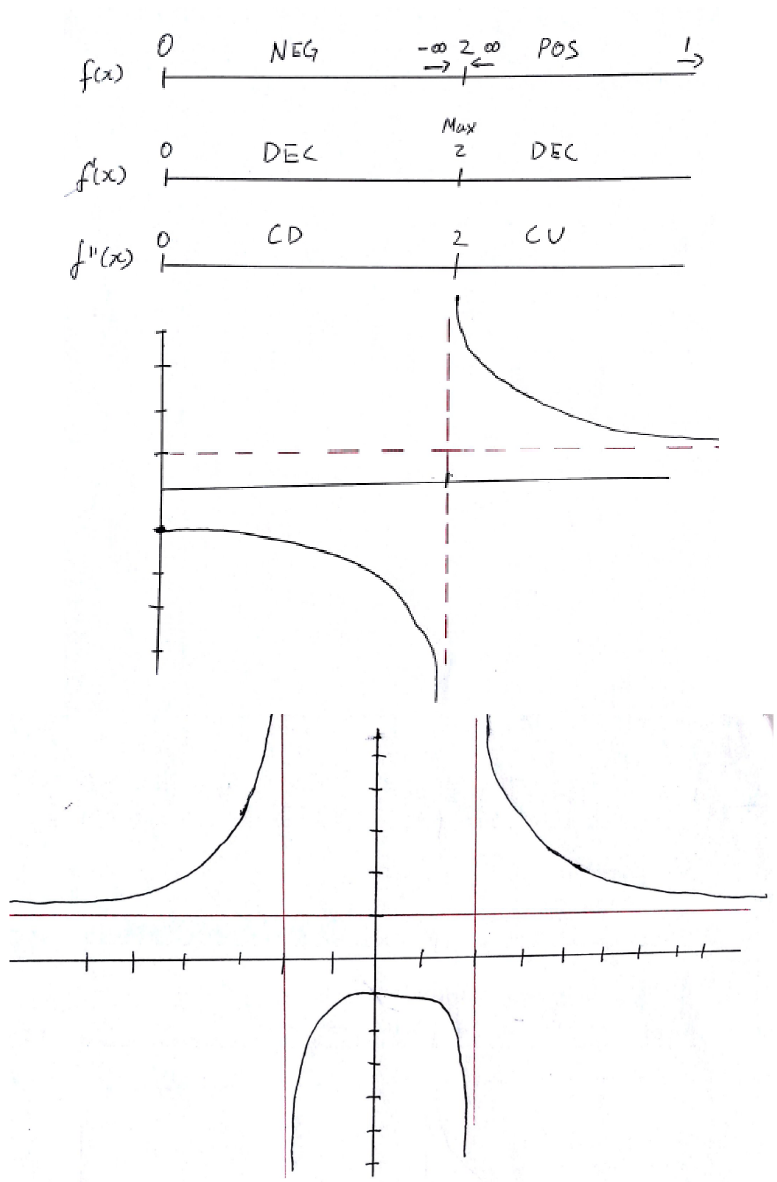
$$\begin{aligned} f''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{-16x}{\left(x^2-4\right)^2} \\ &= \frac{-16\left(x^2-4\right)^2+64x^2\left(x^2-4\right)}{\left(x^2-4\right)^4} \\ &= \frac{-16\left(x^2-4\right)+64x^2}{\left(x^2-4\right)^3} \\ &= \frac{16\left(3x^2+4\right)}{\left(x^2-4\right)^3} \end{aligned}$$

We can look at the numerator to see that we won't get any additional points of interest. Similarly, the denominator does not give us any new points of interest. So we can just partition according to the discontinuities:

Interval	$f''(x)$	CU/CD
$(0,2)$	$f''(1)<0$	CD
$(2,\infty)$	$f''(4)>0$	CU

We can also check the $x=0$ is a local maximum by using the second derivative test. We check that $f''(0)<0$, so it is concave down and hence must be a local maximum.

10. Combining the information, we can sketch the graph for positive x values and then use symmetry to get the other half.



(b)

$$g(x) = (x^2 - 4x)e^{-x}.$$

Solution: Rinse and repeat. Let just go through all the steps:

1. **Domain:** We have a polynomial multiplied by an exponential function. Since both are defined on all the real numbers we have that $D = \mathbb{R}$

2. **Symmetry:** We evaluate:

$$\begin{aligned} f(-x) &= ((-x)^2 - 4(-x))e^{-(-x)} \\ &= (x^2 + 4x)e^x \end{aligned}$$

What we obtain is not $f(x)$ nor $-f(x)$. So this function is not odd and not even. (Sigh, no shortcuts here).

3. **y-intercept:** Evaluate $f(0)$ to get $f(0) = (0^2 - 4(0))e^0 = 0$.

4. **x-intercept:** Similarly, we solve for $f(x) = 0$. Since $e^{-x} > 0$, any roots must have come from $x^2 - 4x = x(x-4)$. Our roots are therefore $x = 0, 4$.

5. **Sign:** Partitioning our domain by the roots, we get:

Interval	$f(x)$	POS/NEG
$(-\infty, 0)$	$f(-1) > 0$	POS
$(0, 4)$	$f(1) < 0$	NEG
$(4, \infty)$	$f(5) > 0$	POS

6. **Horizontal Asymptote:** To check for horizontal asymptotes, we take the limits as x goes to $\pm\infty$. Taking the negative case first (because its easier), we define $u = -x$ to get:

$$\lim_{x\to-\infty}(x^2 - 4x)e^{-x} = \lim_{u\to\infty}(u^2 + 4u)e^u.$$

This limit composes of two functions that each tend to infinity as x gets really big. So their product must also be really big. Hence the limit must be ∞. Now to the positive side:

$$\lim_{x\to\infty}(x^2 - 4x)e^{-x} = \lim_{x\to\infty}\frac{x^2 - 4x}{e^x}$$

which we can tackle using L'Hopital's rule since direct substitution gives "$\frac{\infty}{\infty}$":

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}x^2 - 4x}{\frac{\mathrm{d}}{\mathrm{d}x}e^x}$$
$$= \lim_{x\to\infty}\frac{2x-4}{e^x}$$

We will need L'Hopital's rule again due to the indeterminate form

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}2x - 4}{\frac{\mathrm{d}}{\mathrm{d}x}e^x}$$
$$= \lim_{x\to\infty}\frac{2}{e^x} \qquad = 0$$

So this means we have an horizontal asymptote at $y = 0$.

7. **Vertical Asymptote:** Since we have no discontinuities in the domain, we won't expect any vertical asymptotes.

8. **Intervals of INC/DEC:** To get the intervals of increase and decrease, we need to find the derivative and critical points.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 4x)e^{-x} = (2x-4)e^{-x} - (x^2 - 4x)e^{-x}$$
$$-e^{-x}(x^2 - 6x + 4)$$

So that means the critical points are when $x^2 - 6x + 4 = 0$. Using the quadratic formula, we get:

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 4(1)(4)}}{2} \\ &= \frac{6 \pm \sqrt{20}}{2} \\ &= 3 \pm \sqrt{5} \end{aligned}$$

So the critical points are at $x = 3 - \sqrt{5}, 3 + \sqrt{5}$. Partitioning the interval, we get:

Interval	$f'(x)$	INC/DEC
$(-\infty, 3 - \sqrt{5})$	$f'(0) < 0$	DEC
$(3 - \sqrt{5}, 3 + \sqrt{5})$	$f'(2) > 0$	INC
$(3 + \sqrt{5}, \infty)$	$f'(10) < 0$	DEC

Using this information, we can see that the point $x = 3 - \sqrt{5}$ is a local minimum cause the function goes from decreasing to increasing. Similarly, the point $x = 3 + \sqrt{5}$ is a local maximum since the function changes from increasing to decreasing at that point.

9. **Intervals of Concavity:** To find the intervals of concavity, we want to partition the function according to the points of interest. So we need to find the second derivative:

$$\begin{aligned} f''(x) &= \frac{\mathrm{d}}{\mathrm{d}x} - e^{-x}(x^2 - 6x + 4) \\ &= e^{-x}(x^2 - 6x + 4) - e^{-x}(2x - 6) \\ &= e^{-x}(x^2 - 8x + 10) \end{aligned}$$

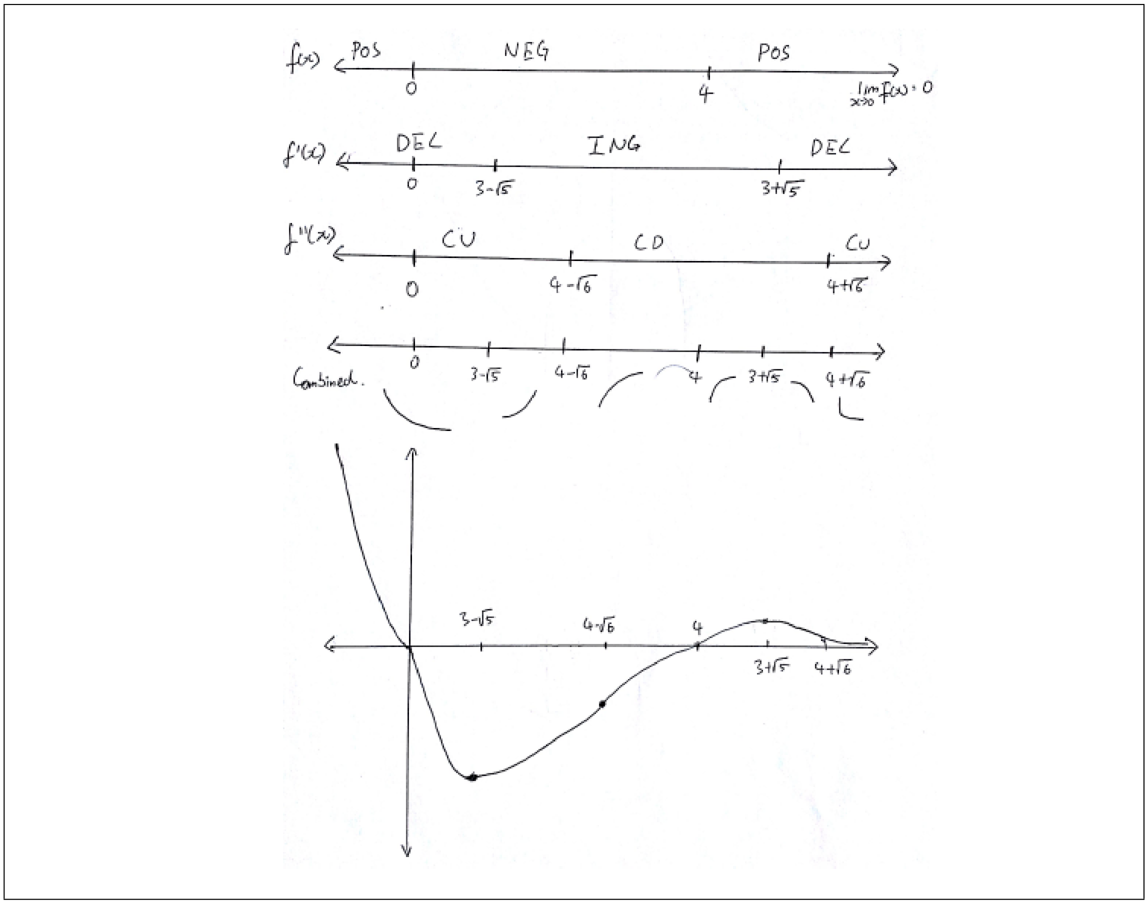
So our points of interests are when $x^2 - 8x + 10 = 0$.

$$\begin{aligned} x &= \frac{8 \pm \sqrt{64 - 4(1)(10)}}{2} \\ &= \frac{8 \pm \sqrt{24}}{2} \\ &= 4 \pm \sqrt{6} \end{aligned}$$

Partitioning our domain by these points of interest, we get:

Interval	$f''(x)$	CU/CD
$(-\infty, 4 - \sqrt{6})$	$f'(0) > 0$	CU
$(4 - \sqrt{6}, 4 + \sqrt{6})$	$f'(3) < 0$	CD
$(4 + \sqrt{6}, \infty)$	$f'(10) > 0$	CU

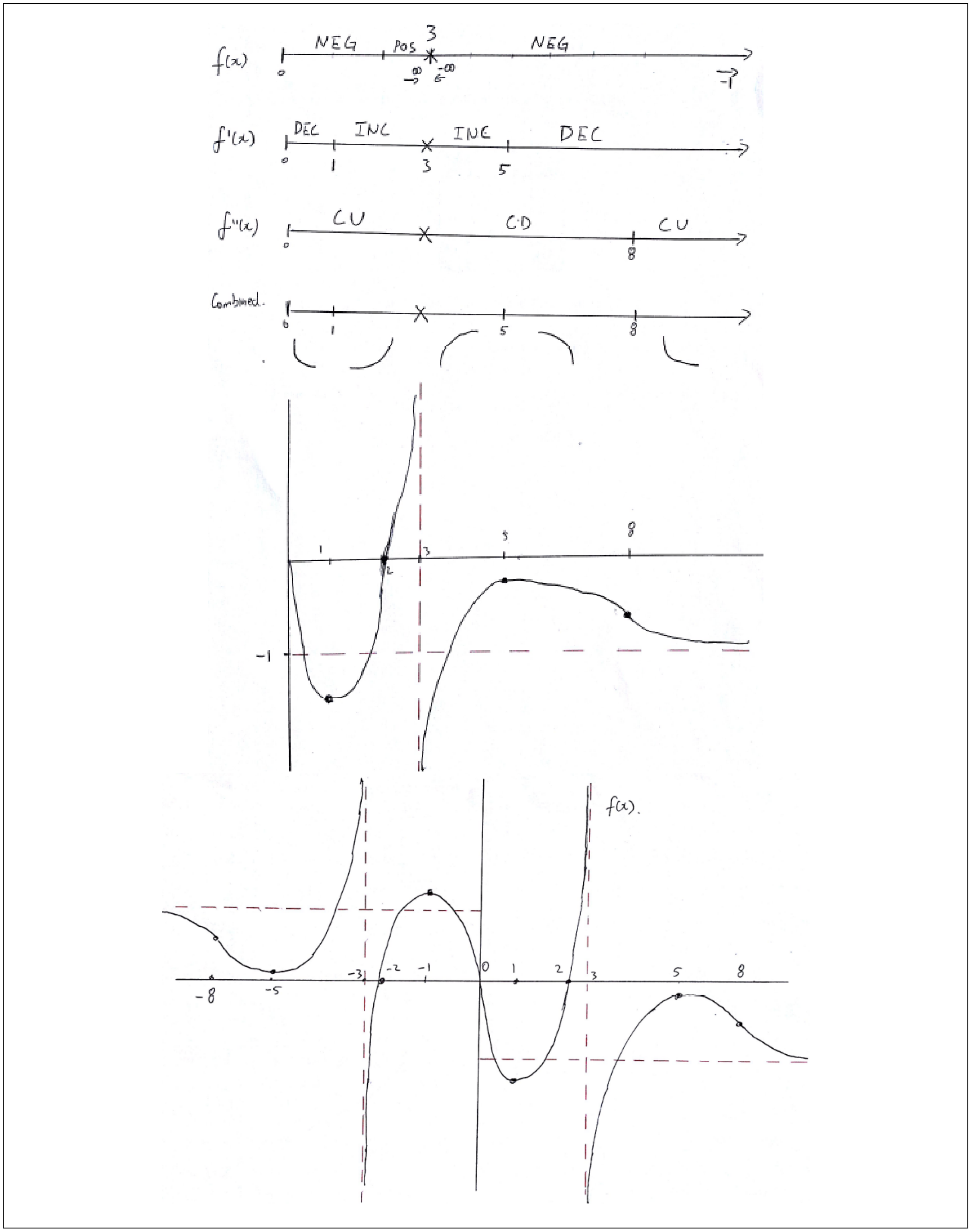
10. Now we just combine the information:



3. Sketch a graph of a function $f(x)$ with the following properties.

- $f(x)$ is an odd function with a root at $x = 0$ and $\lim_{x\to\infty} f(x) = -1$,
- $f(x)$ has the domain: $\mathbb{R} - \{-3, 3\}$,
- $\lim_{x\to 3^-} f(x) = \infty$, $\lim_{x\to 3^+} f(x) = -\infty$,
- $f(x)$ is negative on the interval $(0, 2) \cup (3, \infty)$ and positive on the interval $(2, 3)$,
- $f(x)$ is differentiable on the entire domain with critical points at $x = 1, 5$,
- $f'(x) > 0$ on the interval $(1, 3) \cup (3, 5)$ and $f'(x) < 0$ on $(0, 1) \cup (5, \infty)$,
- The second derivative of $f(x)$ exists on the entire domain,
- $f''(x) > 0$ on the interval $(0, 3) \cup (8, \infty)$ and $f''(x) < 0$ on the interval $(3, 5)$.

Solution: For this question, we should create the three number lines and transfer the information from the question to the number line. Then it is simply a matter of combining that information. For the last item, I intended to write $f''(x) < 0$ on interval $(3, 5)$. But since it is not stated, one can make any assumption they wish on the missing interval. Since the function is odd, we can concentrate on the postive x values and then extend it to the real numbers.



4. [Challenge] Find an algebraic equation satisfying the above conditions. *Hint: A piecewise function is probably most practical. Partition the domain in convenient places.*

Solution: This problem is really tricky, we should partition the function into sensible pieces. I chose to pick points where there are discontinuities as well as points where the derivative is 0. The second condition is due to the fact that every point has to be differentiable, so the derivatives have to match up on both sides. Zero derivatives is probably the easiest choice. By using this partition, we will need four different intervals: $(0,1),(1,3),(3,5),(5,\infty)$. Since there is a discontinuities at $x=3$, we really only have to worry about the functions joining up at $x=1$ and $x=5$.

- Let's do the second interval first: $(1,3)$ first because it contains a vertical asymptote. We need the derivative to be zero when $x=1$ and a vertical asymptote to positive infinity when x gets close to 3. This looks very similar to one of the functions we had in the last questions $\frac{x^2+4}{x^2-4}$ with zero slope at $x=0$ and discontinuity at $x=2$. Now we make use of graph transformations. We need a positive asymptote, so we will need to flip horizontally first. Then shift the graph right by one to give the zero slope at $x=1$ and asymptote at $x=3$. Then we have to shift it so that the point $(2,0)$ is on the curve. This gives:

$$\begin{aligned} f(x) &= -\frac{(x-1)^2+4}{(x-1)^2-4} - \frac{5}{3} \\ &= \frac{-8x(x-2)}{3(x+1)(x-3)} \end{aligned}$$

  We can make our life easier by multiplying by 3 so get rid of fractions in a later step.

$$= \frac{-8x(x-2)}{(x+1)(x-3)}$$

- Now to the first interval: $(0,1)$. We want a function starting at the origin, decreasing and concave up. A simply parabola will do the trick. This is where multiplying by 3 in the previous step makes things easier. From the previous step, we need that $f(1)=-2$. So we can use the turning point formula (if we want).

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &= a(x-1)^2-2 \end{aligned}$$

  The second point is the origin, so we have.

$$\begin{aligned} 0 &= a(-1)^2-2 \\ 2 &= a \end{aligned}$$

  So we have:

$$f(x) = 2(x-1)^2-2$$

- For the third interval, we have a similar situation as the second interval where we have a join a vertical asymptote and a point of zero slope. So we can use the same method. But I will present a second solution. Any function will with a vertical asymptote and critical point will work, so I pick something like $\frac{x^2}{-x-2}$ so that we have the vertical asymptote and critical point at distance two apart with the former occurring before the latter. Then its a matter of shift it to the right by five (since critical point is at $x=5$) and then down by k. I picked $k=0.5$ since we need a decreasing function for the final interval and it has to have a horizontal asymptote at $y = -1$. We end up with:

$$f(x) = \frac{(x-5)^2}{-x+3} - \frac{1}{2}$$

- Final interval: $(5, \infty)$. This last bit is interesting, we need a function with a critical point and a horizontal asymptote. The bell curve is a suitable choice. We can even determine where the concavity changes too. Recall that the bell curve has a critical point at μ and changes concavity at $\mu \pm \sigma$. So that means we want $\mu = 5$ and $\sigma = 3$ giving:

$$f(x) = e^{-\frac{(x-5)^2}{18}}$$

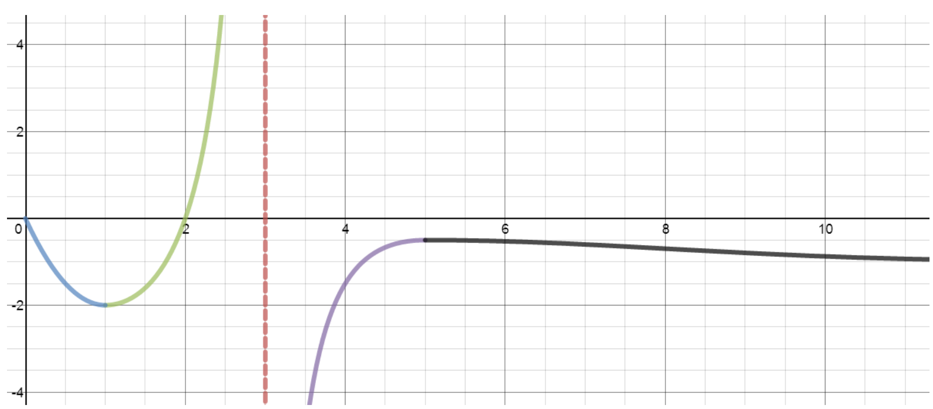
We then deal with the vertical shift and compression. Since the range of this function (From previous assignment is $(0, 1]$), we need to compress it by half and then shift it down by 1 to get the range between $(-1, 0.5]$ as necessary. So our function looks like:

$$f(x) = \frac{1}{2}e^{-\frac{(x-5)^2}{18}} - 1$$

Woah... that was a lot of work. Combining all the functions, we get:

$$f(x) = \begin{cases} 2(x-1)^2 - 2 & x \in (0, 1] \\ \frac{-8x(x-2)}{(x+1)(x-3)} & x \in (1, 3) \\ \frac{(x-5)^2}{-x+3} - \frac{1}{2} & x \in (3, 5] \\ \frac{1}{2}e^{-\frac{(x-5)^2}{18}} - 1 & x \in (5, \infty) \end{cases}$$

Plotting those functions on a graph with the correct intervals, we get:



which looks like (a much better version) of the function we obtained in the previous question. We just need to apply the symmetry to get the full function.