ASSIGNMENT 1·10 (Section 002) Due: Friday, November 8

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions (in full sentences) with complete arguments and justifications in a linear, coherent manner. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. Your work must be your own and must be self-contained. Assignments must be stapled, with your name and student number at the top of each page. The assignment is due at the beginning of class on the due date.

1. Let

$$f(x) = \frac{\sin x \cos x}{\sin x + \cos x}.$$

Find the equation of the line tangent to f through the point $(\frac{\pi}{2}, 0)$.

Solution

The slope of the tangent line through the point $(\frac{\pi}{2}, 0)$ will be $f'(\frac{\pi}{2})$. We compute by using the quotient rule

$$\begin{aligned} f'(x) &= \frac{(\sin x \cos x)' \cdot (\sin x + \cos x) - (\sin x \cos x) \cdot (\sin x + \cos x)'}{(\sin x + \cos x)^2} \\ &= \frac{(\cos^2 x - \sin^2 x) \cdot (\sin x + \cos x) - (\sin x \cos x) \cdot (\cos x - \sin x)}{(\sin x + \cos x)^2} \end{aligned}$$

where we have used product rule for $(\sin x \cos x)'$. We can simplify this expression by expanding

$$\begin{aligned} f'(x) &= \frac{\cos^2 x \sin x + \cos^3 x - \sin^3 x - \cos x \sin^2 x - \sin x \cos^2 x + \sin^2 x \cos x}{\sin^2 x + 2 \sin x \cos x + \cos x^2} \\ &= \frac{\cos^3 x - \sin^3 x}{1 + 2 \sin x \cos x}. \end{aligned}$$

Now, as planned, we compute

$$\begin{aligned} f'\left(\frac{\pi}{2}\right) &= \frac{\cos^3\left(\frac{\pi}{2}\right) - \sin^3\left(\frac{\pi}{2}\right)}{1 + 2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} \\ &= \frac{0-1}{1+0} = -1. \end{aligned}$$

So, the slope of the tangent line passing through point $(\frac{\pi}{2}, 0)$ is -1. We can use this information to write the equation of the tangent line in point-slope form

$$y - 0 = -1(x - \frac{\pi}{2})$$

or

$$y = -1(x - \frac{\pi}{2}).$$

2. Recall that given a function, say f, we can think about its derivative, which we denote $f'(x)$ (providing the function is differentiable). The derivative $f'(x)$ is itself a function and so conceivably we could take its derivative (providing it is also differentiable). We call this new function the second derivative of f and denote it by $f''(x)$. In other words $f''(x) = (f'(x))'$. For each part in this problem make sure you justify your answer.

(a) Find a function $f(x)$ such that

$$f''(x) = -f(x). \tag{1}$$

(b) Find another function satisfying (1).

(c) Find infinitely many functions satisfying (1).

(d) Find a function satisfying (1) with the property that $f(0) = 0$ and $f(\pi/2) = 2$.

Solution

(a) We claim that $f(x) = \sin x$ satisfies (1). Observe $f'(x) = \cos x$ and $f''(x) = -\sin x = -f(x)$.

(b) We can also take $f(x) = \cos x$. Then $f'(x) = -\sin x$ and so $f''(x) = -\cos x = -f(x)$.

(c) Notice that any multiple of $\sin x$ or $\cos x$ will also be a solution of (1). We could take, for example, the functions $n \sin x$ where $n = \{1, 2, 3, \ldots\}$. This gives infinitely many functions satisfying (1). We might also notice that the sum of any two solutions of (1) is also a solution of (1). The most general solution is

$$f(x) = a \sin x + b \cos x, \quad \text{where} \quad a, b \in \mathbb{R}.$$

Observe,

$$\begin{aligned} f'(x) &= a \cos x - b \sin x \\ f''(x) &= -a \sin x - b \cos x = -f(x) \end{aligned}$$

(it turns out that there are no other solutions to our differential equation asside from functions that can be written as above). Notice that if we take $a = b = 0$ we get $f(x) = 0$ which is a solution to (1). We call this solution the *trivial* solution.

(d) Take $f(x) = 2 \sin x$. Then f satisfies (1), as we have seen. As well $f(0) = 2 \sin(0) = 0$ and $f(\pi/2) = 2 \sin(\pi/2) = 2$ as desired.

3. Let

$$f(x) = \begin{cases} (x^2 + 1) \sin x & x < 0 \\ g(x) & 0 \leq x \leq \pi \\ \cos^2 x & x > \pi \end{cases}$$

Find $g(x)$ so that f is differentiable everywhere. Justify your answer.

Solution

If f is to be differentialble it must first be continuous. Note that $(x^2 + 1) \sin x$ and $\cos^2 x$ are both continuous functions (they are compositions of continuous functions). We will also require $g(x)$ to be continous on $(0, \pi)$. Moreoever we need f to be continous across 0 and π. In light of this we get the following conditions

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$$

or rather

$$\lim_{x \to 0^-} (x^2 + 1) \sin x = \lim_{x \to 0^+} g(x) \quad \text{and} \quad \lim_{x \to \pi^-} g(x) = \lim_{x \to \pi^+} \cos^2 x.$$

However, all of these functions are continuous so we can evaluate these limits by direct substitution. This gives us the following two conditions

$$g(0) = 0 \quad \text{and} \quad g(\pi) = 1.$$

With these conditions in place our function f will be continuous (the branches line up so there are no jumps). However, not all continuous functions are differentiable; we need the branches of f to line up in a smooth way (no corners). Function f will be differentiable if $g'(0)$ is equal to the derivative of $(x^2+1)\sin x$ at 0 and $g'(\pi)$ is equal to the derivative of $\cos^2 x$ at π. We compute

$$\big((x^2+1)\sin x\big)' = 2x\sin x + (x^2+1)\cos x$$

which gives 1 when $x = 0$. Hence $g'(0) = 1$. Similarly for the other side

$$\big(\cos^2 x\big)' = (\cos x \cos x)' = -\sin x \cos x - \cos x \sin x$$

which gives 0 when $x = \pi$. Hence $g'(\pi) = 0$. We have shown that any function g that is differentiable on $(0, \pi)$ and satisfies the four conditions

$$g(0) = 0, \quad g(\pi) = 1, \quad g'(0) = 1, \quad g'(\pi) = 0$$

will make f differentiable everywhere. It remains to find such a function (there are infinitely many such functions). We can hunt for a cubic function to do the job. The gerneral form of a cubic polynomial is

$$g(x) = ax^3 + bx^2 + cx + d$$

with derivative

$$g'(x) = 3ax^2 + 2bx + c.$$

First require that

$$0 = g(0) = a(0)^3 + b(0)^2 + c(0) + d$$

this gives $d = 0$. Next we insist that

$$1 = g'(0) = 3a(0)^2 + 2b(0) + c$$

and so $c = 1$. We have narrowed the search to finding a and b. The remaining two conditions give us the following two equations

$$g(\pi) = 1 = a\pi^3 + b\pi^2 + \pi \tag{2}$$

and

$$g'(\pi) = 0 = 3a\pi^2 + 2b\pi + 1. \tag{3}$$

We can use either substitution or elimination to solve this system of two equations with two variables. We will use elimination. Multiply the first equation by 2 and the second by π to see

$$\begin{aligned} 2 - 2\pi &= 2a\pi^3 + 2b\pi^2 \\ -\pi &= 3a\pi^3 + 2b\pi^2. \end{aligned}$$

Subtracting the two yields,

$$2 - \pi = -a\pi^3$$

or

$$a = \frac{\pi - 2}{\pi^3}.$$

With a in hand we can solve for b by substituting back into (2)

$$\begin{aligned} 1 &= \frac{\pi-2}{\pi^3}\pi^3 + b\pi^2 + \pi \\ 1-\pi &= \pi - 2 + b\pi^2 \\ 1-\pi-\pi+2 &= b\pi^2 \\ b &= \frac{3-2\pi}{\pi^2} \end{aligned}$$

So, the polynomial

$$g(x) = \frac{\pi-2}{\pi^3}x^3 + \frac{3-2\pi}{\pi^2}x^2 + x$$

will make f differentiable everywhere.

The procedure we just performed is similar to a technique called cubic spline. This technique uses piecewise cubic functions to interpolate given data points in a 'smooth' way.

If cubic polynomials are not to your liking and you don't mind trying to get a little fancy we can try to patch up f with a trig function. Unfortunately attempts to stretch a sine function fail. The function $g(x) = \sin(x/2)$ fails since $g'(0) = 1/2 \neq 1$. As well, the function $g(x) = 2\sin(x/2)$ fails since $g(\pi) = 2 \neq 1$.

What we can do is use a regular sine function to lift off from 0 and bring us to $\pi/2$ after which we continue onwards with a constant function. The function we are suggesting looks like this

$$g(x) = \begin{cases} \sin x & 0 \leq x \leq \pi/2 \\ 1 & \pi/2 < x \leq \pi \end{cases}$$

Note that $g(x)$ is differentiable on the interval $(0,\pi)$. The only point of contention is $\pi/2$. Observe that $\sin(\pi/2) = 1$ so $g(x)$ is continuous at $\pi/2$. Also the derivative of $\sin x$ at $\pi/2$ is $\cos(\pi/2) = 0$ which equals the derivative of the constant function 1. So $g(x)$ is differentiable at $\pi/2$. We now claim that

$$f(x) = \begin{cases} (x^2+1)\sin x & x < 0 \\ g(x) & 0 \leq x \leq \pi \\ \cos^2 x & x > \pi \end{cases}$$

$$= \begin{cases} (x^2+1)\sin x & x < 0 \\ \sin x & 0 \leq x \leq \pi/2 \\ 1 & \pi/2 < x \leq \pi \\ \cos^2 x & x > \pi \end{cases}$$

Is differentiable everywhere. This is the case since g satisfies the four conditions we derived earlier. That is

$$\begin{aligned} g(0) &= \sin(0) = 0 \\ g(\pi) &= 1 \\ g'(0) &= \cos(0) = 1 \\ g'(\pi) &= 0. \end{aligned}$$

Hence we have found another $g(x)$ that will do the job.