1. Let $f(x) = x^n e^x$. Find all the values of n for which $f(x)$ satisfies the differential equation
$$f'''(x) - 3f''(x) + 3f'(x) - f(x) = 0.$$

$f(x) = x^n e^x$

$f'(x) = n x^{n-1} e^x + x^n e^x$

$f''(x) = n(n-1) x^{n-2} e^x + n x^{n-1} e^x + n x^{n-1} e^x + x^n e^x$

$= n(n-1) x^{n-2} e^x + 2n x^{n-1} e^x + x^n e^x$

$f'''(x) = n(n-1)(n-2) x^{n-3} e^x + n(n-1) x^{n-2} e^x + 2n(n-1) x^{n-2} e^x + 2n x^{n-1} e^x + n x^{n-1} e^x + x^n e^x$

$= n(n-1)(n-2) x^{n-3} e^x + 3n(n-1) x^{n-2} e^x + 3n x^{n-1} e^x + x^n e^x$

$f'''(x) - 3f''(x) + 3f'(x) - f(x) = n(n-1)(n-2) x^{n-3} e^x$

For this expression to be zero for any value of x we must have $n(n-1)(n-2) = 0$, or, equivalently $\boxed{n=0 \quad \text{or} \quad n=1 \quad \text{or} \quad n=2}$.

2. For $c > 0$ let $f(c)$ be defined as
$$f(c) = \lim_{h \to 0} \frac{c^h - 1}{h}.$$

(a) Find $f(1)$.

$f(1) = \lim_{h \to 0} \frac{1^h - 1}{h} = 0$

(b) Find the value of $f(2)$ with three decimal places. To do so, use your calculator to find the values of $\frac{2^h - 1}{h}$ for increasingly smaller and smaller values of h.

$f(2) = \lim_{h \to 0} \frac{2^h - 1}{h}$

h	$(2^h - 1)/h$
0.1	0.7177...
0.01	0.6955...
0.001	0.6933...
0.0001	0.6931...
−0.0001	0.6931...

so $f(2) = 0.693\ldots$

(c) Find the value of $f(4)$ with three decimal places using a similar computation.

$f(4) = \lim_{h \to 0} \frac{4^h - 1}{h}$

h	$(4^h - 1)/h$
0.01	1.3959...
0.001	1.3872...
0.0001	1.3863...
0.00001	1.3863...
−0.00001	1.3862...

So $f(4) = 1.386\ldots$

(d) Show that for $c > 0$ and $d > 0$, the equality $f(c \cdot d) = f(c) + f(d)$ holds.
Hint: you might find the following identity useful for this purpose: $(cd)^h - 1 = (c^h - 1)d^h + (d^h - 1)$.

$$f(c \cdot d) = \lim_{h \to 0} \frac{(c \cdot d)^h - 1}{h} = \lim_{h \to 0} \frac{(c^h - 1)d^h + (d^h - 1)}{h} =$$

$$= \lim_{h \to 0} \frac{c^h - 1}{h} \cdot d^h + \lim_{h \to 0} \frac{d^h - 1}{h} = \lim_{h \to 0} \frac{c^h - 1}{h} \cdot \lim_{h \to 0} d^h + \lim_{h \to 0} \frac{d^h - 1}{h} =$$

$$= f(c) \cdot 1 + f(d) =$$

$$= f(c) + f(d)$$

(e) We have recently talked about another function $g(c)$ defined for $c > 0$ and satisfying $g(cd) = g(c) + g(d)$ for any $c > 0$ and $d > 0$. What is this function?

(f) Can you guess an algebraic expression for $f(c)$ (i.e. an expression that does not involve limits)? Check that this expression agrees with the values of $f(1)$, $f(2)$ and $f(4)$ which you found before.

e) $g(c) = \ln(c)$ satisfies $\ln(c \cdot d) = \ln(c) + \ln(d)$

f) $f(c) = \ln(c)$: $\ln(1) = 0$, $\ln(2) = 0.693\ldots$, $\ln(4) = 1.386\ldots$

3. This question will be based on data in the Wikipedia article "List of cities in the European Union with more than 100,000 inhabitants" located at
http://en.wikipedia.org/wiki/List_of_cities_in_the_European_Union_with_more_than_100,000_inhabitants

(a) Let $N(P)$ be the number of cities in the list with population at least P, where P is measured in hundred thousands of people. For instance $N(1)$ is the number of cities with population at least $100,000$.
For $P = 1$, $P = 2$, $P = 4$, $P = 8$, $P = 16$, $P = 32$ and $P = 64$ find $N, \ln P$ and $\ln N$. Organize your answers in the following table.

P	N	$\ln P$	$\ln N$
1	453	0	6.12
2	181	0.69	5.20
4	68	1.34	4.22
8	25	2.08	3.23
16	11	2.78	2.40
32	3	3.47	1.10
64	1	4.16	0

(b) A hypothesis has been proposed that N and P might be related by a formula of the form

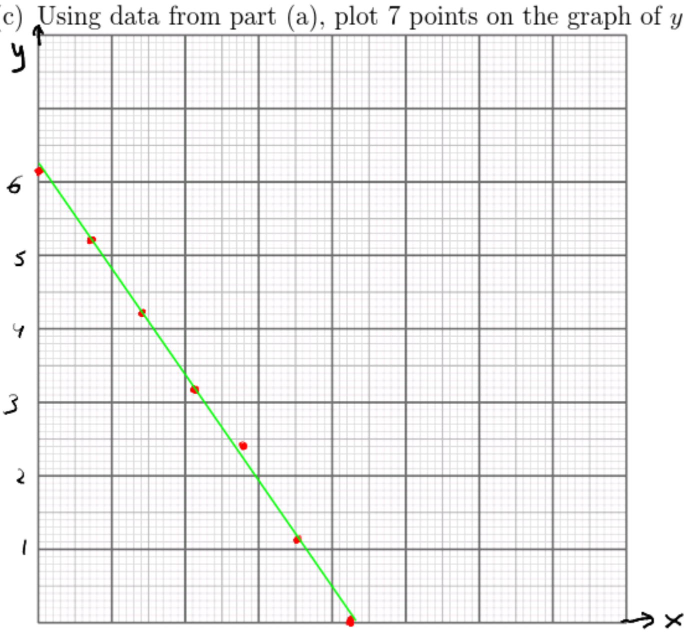
$$N = C \cdot P^m$$

for some C and m.
Let $y = \ln N$ and $x = \ln P$. Express y in terms of x assuming that $N = C \cdot P^m$.

$$\ln N = \ln(C \cdot P^m) = \ln C + m \cdot \ln P$$

so $y = m \cdot x + \ln C$

(c) Using data from part (a), plot 7 points on the graph of y as a function of x.



(d) Using the graph from previous part, estimate the values of m and C.

The green line in the graph has slope $\approx -\frac{6.3}{4.2} \approx -1.47 \approx -1.5$ and y-intercept ≈ 6.3

So a good guess for m would be $-\frac{3}{2}$ and

$$C = e^{6.3} \approx 534$$

So $N = 534 \cdot P^{-3/2}$ is our guess.

Notice how this formula works for different values of P:

P	N	$534 \cdot P^{-3/2}$	percentage error $= \left\vert \frac{N - 534 \cdot P^{-3/2}}{N} \cdot 100\% \right\vert$	over/under estimate
1	453	534	18 %	over
2	181	188	4 %	over
4	68	67	2 %	under
8	25	24	6 %	under
16	11	8	25 %	under
32	3	2.95	2 %	under
64	1	1.04	4 %	over

So it seems that the formula is accurate to within $\pm 25\%$.