# ASSIGNMENT 1· 12-Solutions

*There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.*

1. Let $g(x) = (f(-\cos(x)))^3$. Assume the following values for $f(x)$ and $f'(x)$ given in the following table.

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| $-\frac{\sqrt{2}}{2}$ | 0 | 2 |
| $-\frac{1}{2}$ | -7 | -9 |
| $-\frac{\sqrt{3}}{2}$ | 2 | $-\sqrt{5}$ |

Find the equation of the tangent line to $g(x)$ at $x = \dfrac{\pi}{6}$.

**Solution:** Now, I'm asked to determine the tangent line to $g(x)$ at $x = \frac{\pi}{6}$. In order to find the equation of the tangent line, I'll need the slope of the tangent line at the given point. So, I'll need the value of $g'(x)$ at $x = \frac{\pi}{6}$. We first need to recognize that this is a composition of three functions, with $x^3$ being the outer function, $f(x)$ the middle one, and $-\cos(x)$ the inner function. So, in order to differentiate $g(x)$, we need to use the chain rule. So, we have:

$$g'(x) = 3(f(-\cos(x)))^2 f'(-\cos(x))\sin(x)$$

Now, I want $g'(\frac{\pi}{6})$. We can evaluate this directly and get:

$$\begin{aligned} g'(\frac{\pi}{6}) &= 3(f(-\cos(\frac{\pi}{6})))^2 f'(-\cos(\frac{\pi}{6}))\sin(\frac{\pi}{6}) \\ &= \frac{3}{2} f(-\frac{\sqrt{3}}{2})^2 f'(-\frac{\sqrt{3}}{2}) \end{aligned}$$

In order to get a value for $g'(\frac{\pi}{6})$, we need values of $f$ and $f'$ at $-\frac{\sqrt{3}}{2}$. Thankfully, these are given to us in the table! $f(\frac{-\sqrt{3}}{2}) = 2$ and $f'(\frac{-\sqrt{3}}{2}) = -\sqrt{5}$. Substituting these values back into the last equation, we have:

$$\begin{aligned} g'(\frac{-\sqrt{3}}{2}) &= \frac{3}{2} f(-\frac{\sqrt{3}}{2})^2 f'(-\frac{\sqrt{3}}{2}) \\ &= -\frac{3}{2}(2)^2\sqrt{5} \\ &= -6\sqrt{5} \end{aligned}$$

So, the slope of the tangent line to $g(x)$ at $x = \frac{\pi}{6}$ is $-6\sqrt{5}$. Remember, the equation of the tangent line at $(a, g(a))$ is given by:

$$y - g(a) = g'(a)(x - a)$$

To wrap things up, we need $g(\frac{\pi}{6})$. We'll use our handy table once again.

$$\begin{aligned} g(\frac{\pi}{6}) &= (f(-\cos(\frac{\pi}{6})))^3 \\ &= (f(-\frac{\sqrt{3}}{2}))^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

Hence, the equation of the tangent line is:

$$y - 8 = -6\sqrt{5}(x - \tfrac{\pi}{6})$$

2. It's my birthday and I want an inflatable bouncy castle at my party. The volume of the bouncy castle after inflating it for $t$ minutes is given by the function:

$$V(t) = \frac{e^{\frac{t}{10}}}{t^2 + 1}$$

The height of the bouncy castle in inches is given by the function:

$$h(V) = \sqrt[3]{V + 2}$$

The party is about to start and I'm worried about how long the structure is going to take to fully inflate. Calculuate the instantaneous rate of change of the height with respect to time at $t = 30$ minutes.

**Solution:** We need a function which represents the change in height with time. At this point, we only have change in volume with respect to time and change in height with respect to volume. But, if we take their composition $h \circ V = h(V(t))$, this is now a function who's input is time and output is height. This is exactly what we wanted!
So, let's compute $h \circ V(t)$.

$$\begin{aligned} h \circ V(t) &= h(V(t)) \\ &= h(\frac{e^{\frac{t}{10}}}{t^2+1}) \\ &= \sqrt[3]{\frac{e^{\frac{t}{10}}}{t^2+1} + 2} \end{aligned}$$

We are asked to determine the instantaneous rate of change of height with time at 30 minutes, so we need to determine the derivative of our composition and evaluate it at $t = 30$.

$$\begin{aligned} \frac{d}{dt} h \circ V(t) &= h'(V(t))V'(t) \\ &= \frac{1}{3}\left(\frac{e^{\frac{t}{10}}}{t^2+1} + 2\right)^{-2/3}\left(\frac{e^{\frac{t}{10}}}{t^2+1} + 2\right)' && \text{By the chain rule} \\ &= \frac{1}{3}\left(\frac{e^{\frac{t}{10}}}{t^2+1} + 2\right)^{-2/3}\left(\frac{(t^2+1)(\frac{1}{10}e^{\frac{t}{10}}) - e^{\frac{t}{10}}(2t)}{(t^2+1)^2}\right) && \text{By the quotient rule} \end{aligned}$$

So, the instantaneous rate of change with respect to time is given by:

$$\frac{d}{dt}h\circ V(t)=\frac{1}{3}\left(\frac{e^{\frac{t}{10}}}{t^2+1}+2\right)^{-2/3}\left(\frac{(t^2+1)(\frac{1}{10}e^{\frac{t}{10}})-2te^{\frac{t}{10}}}{(t^2+1)^2}\right)$$

Substituting t=30 gives the desired result of:

$\frac{1}{3}\left(\frac{e^{\frac{30}{10}}}{(30)^2+1}+2\right)^{-2/3}\left(\frac{((30)^2+1)(\frac{1}{10}e^{\frac{30}{10}})-e^{\frac{30}{10}}(60)}{(30^2+1)^2}\right)\approx 0.000155232$

Looks like the hight is barely changing. Something must be wrong with the air flow.

3. The following functions play an important role in the study of differential equations. In particular, they are used to study heat transfer over metal surfaces.
   (a) If $n$ is a positive integer, prove that $\frac{d}{dx}(\sin^n(x)\cos(nx))=n\sin^{n-1}(x)\cos((n+1)x)$.
   (b) Find a similar formula for $\frac{d}{dx}(\cos^n(x)\cos(nx))$.
   (c) Bonus: Find $\frac{d}{dn}(\sin^n(x)\cos(nx))$. Yes, *differentiate the function with respect to $n$.*

   *Hint:* You may find the identity $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ useful for part (a). For part (b), consider using $\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$.

   **Solution:** (a) This is a product of two functions, so we'll need to use the product rule to start things off.

$$\begin{aligned}
\frac{d}{dx}(\sin^n(x)\cos(nx)) &= \sin^n(x)(\cos(nx))'+\cos(nx)(\sin^n(x))' && \text{By the product rule.}\\
&= \sin^n(x)(-n\sin(nx))+\cos(nx)(n\sin^{n-1}(x)\cos(x)) && \text{By the chain rule.}\\
&= n\sin^{n-1}(x)(\sin(x)\sin(nx)+\cos(nx)\cos(x)) && \text{Factor out } \sin^{n-1}(x) \text{ and } n
\end{aligned}$$

Now, the term leftover in parentheses looks very similar to the identity in the hint! In fact, it is exactly what the hint says, but with $x=x$ and $y=nx$. Substituing in the identity we have:

$$\begin{aligned}
\frac{d}{dx}(\sin^n(x)\cos(nx)) &= n\sin^{n-1}(x)(\sin(x)\sin(nx)+\cos(nx)\cos(x))\\
&= n\sin^{n-1}(x)\cos(x+nx)\\
&= n\sin^{n-1}(x)\cos((n+1)x)
\end{aligned}$$

Giving us the desired result!

(b) The same process goes for this part:

$$\begin{aligned}
\frac{d}{dx}(\cos^n(x)\cos(nx)) &= \cos^n(x)(\cos(nx))'+\cos(nx)(\cos^n(x))' && \text{By the product rule.}\\
&= \cos^n(x)(-n\sin(nx))+\cos(nx)(n\cos^{n-1}(x)(-\sin(x))) && \text{By the chain rule.}\\
&= -n\cos^{n-1}(x)(\cos(x)\sin(nx)+\sin(x)\cos(nx)) && \text{Factor out } \cos^{n-1}(x) \text{ and } -n
\end{aligned}$$

Again, this looks very similar to the hint!! We can use the second hint on the term in parentheses. So,

$$
\begin{aligned}
\frac{d}{dx}(\cos^n(x)\cos(nx)) &= -n\cos^{n-1}(x)(\cos(x)\sin(nx)+\cos(nx)\sin(x)) \\
&= -n\cos^{n-1}(x)\sin(x+nx) \\
&= -n\cos^{n-1}(x)\sin((n+1)x)
\end{aligned}
$$

Looks pretty good!! :)

Bonus: What we need to realize here is that we are differentiating with respect to $n$. This means, we want to see the change in the function as n changes. When we differentiated with respect to $x$, $x$ is changing and we didn't care about $n$; it was just some constant. Now, we care about the change in $n$ and not $x$. $x$ is our constant now! So, since $x$ is a constant, so is $\sin(x)$. We have:

$$
\begin{aligned}
\frac{d}{dn}(\sin^n(x)\cos(nx)) &= (\sin(x))^n\cos(nx) \\
&= (\sin(x))^n(\cos(nx))' + \cos(nx)((\sin(x))^n)' && \text{By the product rule} \\
&= (\sin(x))^n(-x\sin(nx)) + \cos(nx)(\sin(x))^n\ln(\sin(x)) && \text{By the chain rule} \\
&= -x\sin^n(x)\sin(nx) + \cos(nx)\sin^n(x)\ln(\sin(x))
\end{aligned}
$$

The important thing to realize is that $x$ is now constant. So, instead of using the power rule on $(\sin(x))^n$, we have an *exponential equation* and need to use that rule for differentiation.