Instructions:

- There are two parts to this assignment. The first part is on WeBWorK.

- The second parts consists of questions on this page.

- This assignment **does not** have to be submitted.

- Solutions will be posted at the end of the week.

- You are encouraged to attempt these problems as this content will be covered in the exam without a formal assignment for it.

Questions:

1. **The Lemniscate of Bernoulli:** Consider the curve given by

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$

(a) Find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ using any technique covered in class.

Solution: In theory, we can isolate for y and then explicitly find the derivative. But in this case, we can see that it might be difficult to accomplish. So we instead go down the implicit differentiation path. Differentiating the left hand side gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2\right)^2 = 2\left(x^2 + y^2\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2\right) \qquad \text{Chain Rule}$$

$$= 2\left(x^2 + y^2\right) \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) \qquad \text{Addition Rule}$$

Differentiating the right hand side gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2a^2(x^2 - y^2)\right) = 2a^2\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - y^2\right) \qquad \text{Constant Multiple}$$

$$= 2a^2\left(2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Combining the two sides give:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(2a^2(x^2 - y^2)\right)$$

$$2\left(x^2 + y^2\right) \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2a^2\left(2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

We can then expand both sides and rearrange to solve of $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$2\left(x^2 + y^2\right) \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2a^2\left(2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\left(x^2 + y^2\right) \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = a^2\left(2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) \qquad \text{Cancel 2}$$

$$2x\left(x^2 + y^2\right) + 2y\left(x^2 + y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2xa^2 - 2ya^2\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{Expand LHS}$$

$$2ya^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\left(x^2 + y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2xa^2 - 2x\left(x^2 + y^2\right) \qquad \text{Rearrange for } \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}2y\left(a^2 + x^2 + y^2\right) = 2x\left(a^2 - x^2 - y^2\right) \qquad \text{Factor LHS}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x\left(a^2 - x^2 - y^2\right)}{2y\left(a^2 + x^2 + y^2\right)} \qquad \text{Divide through}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\left(a^2 - x^2 - y^2\right)}{y\left(a^2 + x^2 + y^2\right)} \qquad \text{Simplify}$$

(b) Fix $a = 2.5$. Find the equation of the tangent line passing through the point $(3, 1)$.

Solution: First, we should check that the point $(3, 1)$ is on the curve. Substituting $a = 2.5$, we get:

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$
$$(x^2 + y^2)^2 = 2(2.5)^2(x^2 - y^2)$$
$$(x^2 + y^2)^2 = 12.5(x^2 - y^2)$$

Substitute the point.

$$((3)^2 + (1)^2)^2 = 12.5((3)^2 - (1)^2)$$
$$(9 + 1)^2 = 12.5(9 - 1)$$
$$(10)^2 = 12.5(8)$$
$$100 = 100$$

Check. The two sides are equal and hence the point is on the curve. Now, to find the tangent line, we make use of the derivative from the previous part to get the slope of

the tangent line at the point $(3, 1)$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x\left(a^2 - x^2 - y^2\right)}{y\left(a^2 + x^2 + y^2\right)} \\
&= \frac{(3)\left((2.5)^2 - (3)^2 - (1)^2\right)}{(1)\left((2.5)^2 + (3)^2 + (1)^2\right)} \\
&= \frac{(3)(6.25 - 9 - 1)}{(1)(6.25 + 9 + 1)} \\
&= \frac{(3)(-3.75)}{(1)(16.25)} \\
&= \frac{-11.25}{16.25} \\
&= -\frac{9}{13}
\end{aligned}$$

Now we can make use of the point slope formula:

$$y - y_1 = m(x - x_1)$$

and insert the point we are given along with the slope we found:

$$\begin{aligned}
y - 1 &= -\frac{9}{13}(x - 3) \\
y &= 1 - \frac{9}{13}(x - 3) \\
y &= -\frac{9}{13}x + \frac{40}{3}
\end{aligned}$$

So the tangent line of the curve at the point $(3, 1)$ is given by:

$$y = -\frac{9}{13}x + \frac{40}{3}$$

(c) Plot a graph of the curve and the tangent line from the previous part (possibly with the aid of a computer program).

Solution: A plot of the curve is below in red. The tangent line at the point $(3, 1)$ is highlighted in green.
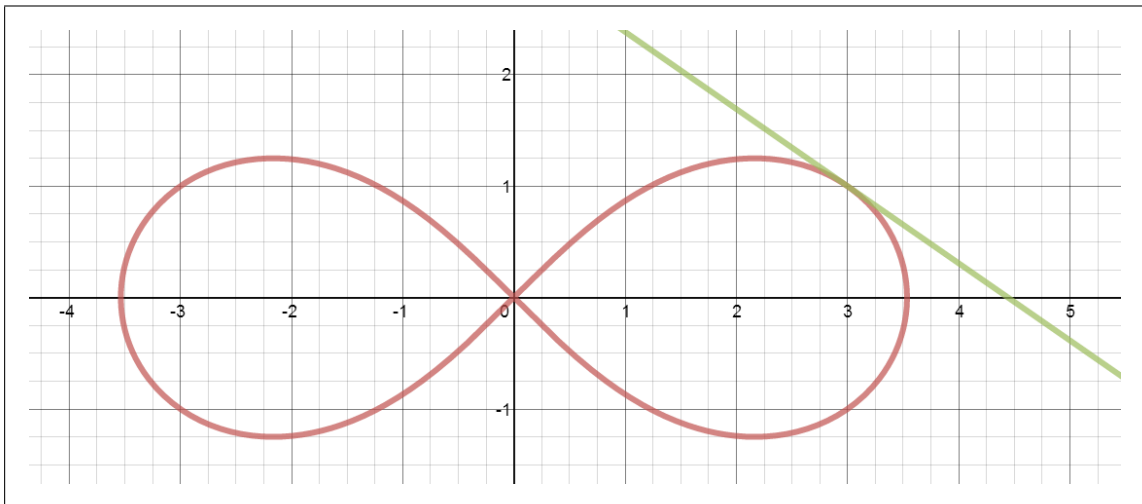
(d) Using the graph, describe what happens if we try to find the tangent line a the origin. How is that reflected in the derivative we've found in the previous part?

Solution: From the graph, we can see that the curve crosses itself at the origin. So that means we would not expect there to be an unique tangent line at the origin. Indeed, we can see that at $(0,0)$, the derivative cannnot be computed simply since we end up with '$\frac{0}{0}$'. With a little bit more work, we can show that the derivative of the curve at that point has to be ± 1 (which also confirms that there is no unique tangent line).

2. **The pitfalls of implicit differentiation:** Consider the following equation:

$$\frac{4(x-y)}{x+y} = \frac{x}{y} + 1$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ using implicit differentiation. *Hint: Fractions are ugly since they imply quotient rule. Simplify first.*

Solution: As the hint suggests, lets simplify first:

$$\frac{4(x-y)}{x+y} = \frac{x}{y} + 1$$
$$\frac{4(x-y)}{x+y} = \frac{x}{y} + \frac{y}{y}$$
$$\frac{4(x-y)}{x+y} = \frac{x+y}{y}$$
$$4y(x-y) = (x+y)^2$$
$$4xy - 4y^2 = x^2 + 2xy + y^2$$
$$0 = x^2 - 2xy + 5y^2$$

Now we can isolate for y, but we see that it is a quadratic in y, which means we will need the quadratic formula and hence quotients and square roots. Both of which are

ugly and we can do without. So lets just implicit differentiate.

$$\frac{\mathrm{d}}{\mathrm{d}x}(0) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - 2xy + 5y^2\right)$$

$$0 = 2x - 2\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + 1\cdot y\right) + 5y\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{Product, Chain rule}$$

$$0 = 2x - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 5y\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{Expand}$$

$$0 = 2x - 2y - (2x - 5y)\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{Factor}$$

$$(2x - 5y)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{2x - 5y}$$

So that means the derivative of the curve is given by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{2x - 5y}$$

(b) Show that there are no points on that curve. That is, there are no possible pair of points $P = (x, y)$ that satisfy the equation.

Solution: Consider the simplified form of the curve:

$$0 = x^2 - 2xy + 5y^2$$
$$0 = (x - y)^2 + 4y^2$$

Note that anything squared is going to be bigger than zero. So that only way the equation can be true is if $y = 0$ and $x - y = 0$. Combined, this gives $x = 0$ and $y = 0$ to be the only potential candidate for a point on the curve. However, in the original form of the statement, we have y in the denominator and hence $y \neq 0$, which means our only candidate cannot be on the curve. This leaves us with no point on the curve. *sad face*.

Note: When we cross multiplied in the first part of the question, we are making the implicit assumption that y and $x + y$ are non-zero. If they are, nothing would work. This highlights the importance of checking whether things makes sense. We can play all the algebraic games we want as per the first part of the question, but if there are no point on the curve, none of what we did in the first part makes any sense.

Note: This highlights an important aspect of implicit differentiation. Without a point on a curve, the derivative we have found is meaningless.

3. **The rise of Android:** In 2010, A popular internet search company revealed 100 million activations for Android superphones. In 2012, the same study revealed 300 million activations for Android superphones. Denote $N(t)$ be the number of activations of Android superphones (in millions) since 2008.

   (a) Justify why it makes sense to use an exponential model to measure $N(t)$?

Solution: It makes sense to measure $N(t)$ using an exponential model. The rate at which new users that hear about (and hence eventually joins) depends on the number of existing users through word of mouth. In addition, the more users, the more revenue for the company and hence more budget for advertising etc.

(b) Use an exponential model to find the number of activations in 2008 (when the first android superphone was released).

Solution: So let us define $t = 0$ to be the year 2008 when Android superphones were released. Then the year 2010 would be $t = 2$ and 2012 would be $t = 4$. So that means we know:

$$N(2) = 100 \qquad\qquad N(4) = 300$$

Using an exponential model, we have that:

$$N(t) = Ae^{rt}$$

which means:

$$N(2) = 100 = Ae^{2r}$$
$$N(4) = 300 = Ae^{4r}$$

Dividing one by the other:

$$\frac{N(4)}{N(2)} = \frac{300}{100} = \frac{Ae^{4r}}{Ae^{2r}}$$
$$3 = e^{4r-2r}$$
$$3 = e^{2r}$$
$$\ln(3) = 2r$$
$$r = \frac{\ln(3)}{2}$$

Now we substitute this into $N(2)$ to get:

$$N(2) = 100 = Ae^{2\left(\frac{\ln(3)}{2}\right)}$$
$$100 = Ae^{\ln(3)}$$
$$100 = 3A$$
$$A = \frac{100}{3}$$

So that means the number of new activations at any time t is given by:

$$N(t) = \frac{100}{3}e^{\frac{\ln(3)}{2}t}$$

which is acceptable. Now substitute $t = 0$ to get:

$$
\begin{aligned}
N(0) &= \frac{100}{3} e^{\frac{\ln(3)}{2} \cdot 0} \\
&= \frac{100}{3} e^{0} \\
&= \frac{100}{3}
\end{aligned}
$$

So the number of new activations in 2008 will be around 33.3 million according to the model.

Aside: We can simplify $N(t)$ further to make our life easier

$$
\begin{aligned}
N(t) &= \frac{100}{3} e^{\frac{t \ln(3)}{2}} \\
&= \frac{100}{3} e^{\ln(3) \cdot \frac{t}{2}} \\
&= \frac{100}{3} \left(e^{\ln(3)} \right)^{\frac{t}{2}} \\
&= \frac{100}{3} \cdot 3^{\frac{t}{2}} \\
&= 100 \cdot 3^{-1} \cdot 3^{\frac{t}{2}} \\
&= 100 \cdot 3^{\frac{t}{2} - 1} \\
N(t) &= 100 \cdot 3^{\frac{t-2}{2}}
\end{aligned}
$$

(c) What will be the projected number of activations for 2015?

Solution: According to our model, the year 2015 will be $t = 7$. So we simply find $N(7)$.

$$
\begin{aligned}
N(t) &= 100 \cdot 3^{\frac{t-2}{2}} \\
N(7) &= 100 \cdot 3^{\frac{7-2}{2}} \\
&= 100 \cdot 3^{2.5} \\
&\approx 1558.85
\end{aligned}
$$

In the year 2015, the model predicts 1.558 Billion new activations.

(d) According to this model, at what point will the number of activations exceed the world population?

Solution: Before we start, it might be worth making a guess as to when that might happen (or if it does). Truth is, this point will happen sooner than you might think. Suppose that at the point it happens, we have 8 billion people in the world (Current, the world population is quoted at 7 billion). So we are interest in the point t when

$N(t) = 8000.$

$$N(t) = \frac{100}{3} e^{\frac{\ln(3)}{2} t}$$

$$8000 = \frac{100}{3} e^{\frac{\ln(3)}{2} t}$$

$$\frac{3 \cdot 8000}{100} = e^{\frac{\ln(3)}{2} t}$$

$$240 = e^{\frac{\ln(3)}{2} t}$$

$$\ln(240) = \frac{\ln(3)}{2} t$$

$$\frac{2 \cdot \ln(240)}{\ln(3)} = t$$

$$t = 9.9773\ldots$$

Recall that $t = 0$ is the year 2008, which means $t = 10$ would be the year 2018. Therefore, the number of new activations will exceed the world population before the year 2018 is over according to this model. Scary huh?

4. **The power of Logs:** Find the equation of the tangent line at $x = 1$ of $g(x)$ defined by:

$$g(x) = \frac{x^7 e^{2x}}{3^{\ln(x)} (x^2 + 5x + 6)^4}$$

Solution: If you're first thought after looking at the function is: one quotient rule, two product rules, and three chain rules, you would be correct. If you're second thought is that your instructor is a cruel sadistic person, then it shows that you probably should be paying more attention in class. :P We have a technique that is perfect for dealing with such functions. Recall that for any differentiable function $f(x)$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(f(x)) = \frac{f'(x)}{f(x)}.$$

We will me making use of this. But first, lets take the natural log of both sides and then simplify:

$$g(x) = \frac{x^7 e^{2x}}{3^{\ln(x)} (x^2 + 5x + 6)^4}$$

$$\ln(g(x)) = \ln \left(\frac{x^7 e^{2x}}{3^{\ln(x)} (x^2 + 5x + 6)^4} \right)$$

$$= \ln \left(x^7 e^{2x} \right) - \ln \left(3^{\ln(x)} (x^2 + 5x + 6)^4 \right)$$

$$= \ln \left(x^7 \right) + \ln \left(e^{2x} \right) - \ln \left(3^{\ln(x)} \right) - \ln \left((x^2 + 5x + 6)^4 \right)$$

$$= 7 \ln (x) + 2x \ln (e) - \ln(x) \ln (3) - 4 \ln \left(x^2 + 5x + 6 \right)$$

$$= 7 \ln (x) + 2x \cdot 1 - \ln (3) \ln(x) - 4 \ln \left(x^2 + 5x + 6 \right)$$

Now we implicit differentiate by applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides and apply the derivative to each term:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(g(x))\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(7\ln(x) + 2x - \ln(3)\ln(x) - 4\ln\left(x^2 + 5x + 6\right)\right)$$

$$\frac{g'(x)}{g(x)} = 7\cdot\frac{1}{x} + 2 - \ln(3)\cdot\frac{1}{x} - 4\cdot\frac{2x+5}{x^2+5x+6}$$

$$g'(x) = g(x)\left(\frac{7}{x} + 2 - \frac{\ln(3)}{x} - 4\cdot\frac{2x+5}{x^2+5x+6}\right)$$

$$g'(x) = \frac{x^7 e^{2x}}{3^{\ln(x)}(x^2+5x+6)^4}\cdot\left(\frac{7}{x} + 2 - \frac{\ln(3)}{x} - 4\cdot\frac{2x+5}{x^2+5x+6}\right)$$

Now that we have the derivative, we can find:

$$g(1) = \frac{e^2}{20736}$$

$$g'(1) = \frac{e^2}{20736}\cdot\left(\frac{20}{3} - \ln(3)\right)$$

Using the point-slope formula, we get:

$$y = g(1) + g'(1)(x-1)$$

$$y = \frac{e^2}{20736} + \frac{e^2}{20736}\cdot\left(\frac{20}{3} - \ln(3)\right)\cdot(x-1)$$

which is indeed a straight line and the tangent line at $x = 1$.