

Questions:

1. We have a function such that

$$\lim_{x \rightarrow 4} f(x) = -3.$$

Justify why or why not: If x_1 is closer to 4 than x_2 , then $f(x_1)$ will be closer to -3 than $f(x_2)$.

To justify why, construct an argument why it must be true. To show why not, it suffices to provide one example where the condition is true but the conclusion is false.

Solution: My claim is that the statement is false. That is, I can come up with a function $f(x)$ that contains a point x_1 that is closer to 4 than x_2 but with $f(x_1)$ being further away than $f(x_2)$. Consider the following function:

$$f(x) = \begin{cases} 1 - x & x \leq 4 \\ 3x - 15 & x > 4 \end{cases}$$

Now suppose we take the points $x_1 = 4.05$ and $x_2 = 3.9$. We have that x_1 is closer to 4 than x_2 . But then we calculate the y -values and we get:

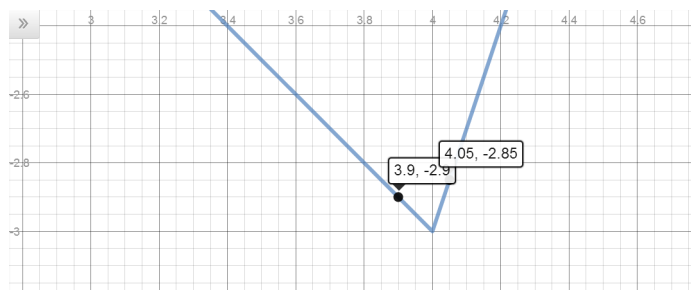
$$\begin{aligned} f(x_1) &= f(4.05) \\ &= 3(4.05) - 15 \\ &= 12.15 - 15 \\ &= -2.85 \end{aligned}$$

and for x_2 , we have:

$$\begin{aligned} f(x_2) &= f(3.9) \\ &= 1 - 3.9 \\ &= -2.9. \end{aligned}$$

This means $f(x_2)$ is closer to -3 than $f(x_1)$ despite x_1 being closer to x_2 . Hence, we have shown that statement is false by providing a counter example.

Some additional details: We can see what's happening by plotting a graph:



It is obvious that one side of the function is going towards the limit much quicker than the other, and that is exactly what we are taking advantage of here. By picking x_1 and x_2 on different sides of 4, we can get the result we want. We can make this even more obvious by taking one of the functions to be constant. If we want to enforce an extra condition that x_1 and x_2 have to be on the same side of 4, we can consider a parabola, say

$f(x) = (x - 4)(x - 3.9) - 3$. Then while $x_1 = 3.95$ is close to 4 than $x_2 = 3.90$, we have $f(3.95)$ being further away from -3 than $f(3.90) = -3$.

Other possible solutions: If we think about this carefully, we can also disprove the claim by considering the constant function or a a function with a small bump at one point. for example:

$$f(x) = \begin{cases} -3 & x \neq 4.01 \\ 10 & x = 4.01 \end{cases}$$

Then we can pick $x_1 = 4.01$ and $x_2 = 4.02$ and get the same conclusion.

2. Construct and sketch the graph of a function $g(t)$ with:

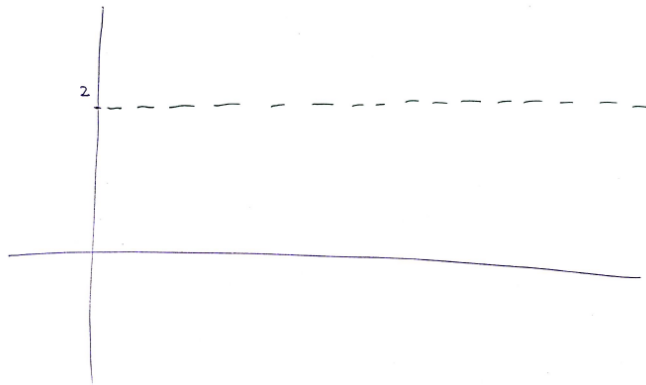
- Domain = \mathbb{R} ,
- For any value of t , $g(t) \neq 2$,

as well as the following properties:

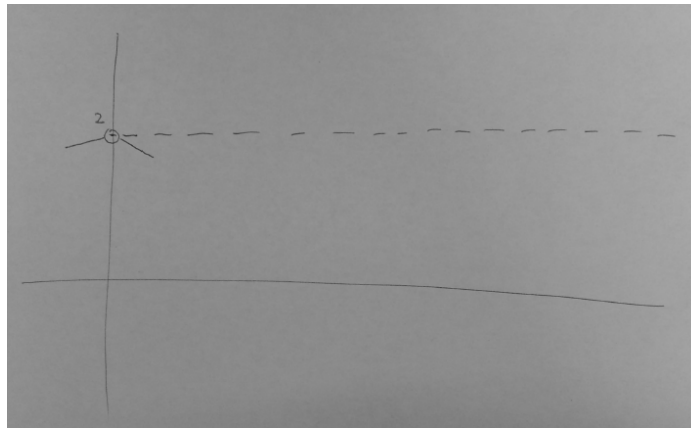
$$\begin{array}{ll} g(0) = g(4) & \lim_{t \rightarrow 0} g(t) = 2 \\ \lim_{t \rightarrow 4^-} g(t) = -1 & \lim_{t \rightarrow 4^+} g(t) = 0 \end{array}$$

For this question and the next one, ‘construct’ means to find an algebraic expression for the function. That is, we want to find: $g(t) = \dots$.

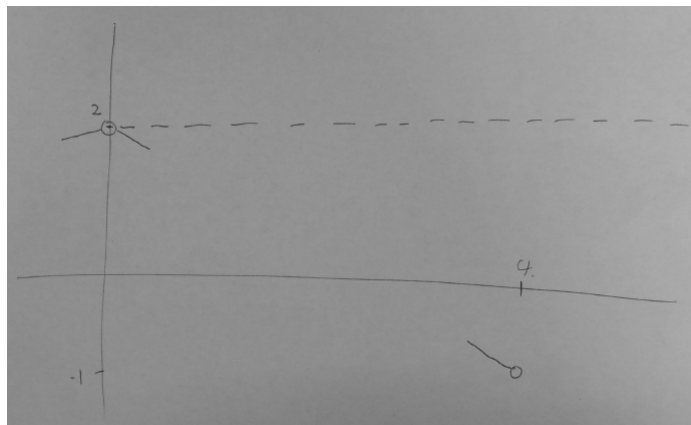
Solution: This is one of those questions where its best to begin with a graph. Let’s check off each of the criteria one by one as we draw the graph. We start by observing that the function cannot cross the line $y = 2$.



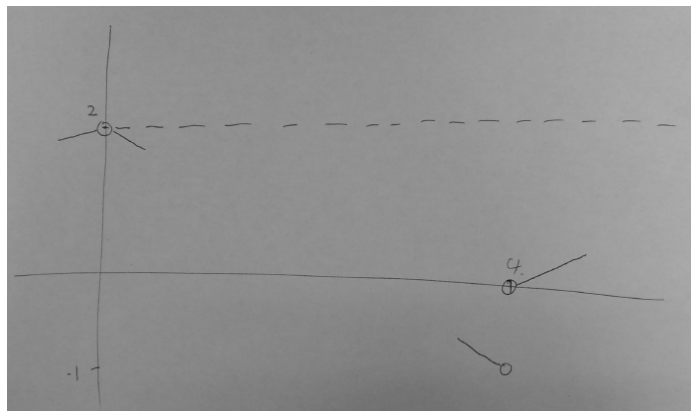
From there, we can add in the condition that $\lim_{t \rightarrow 0} g(t) = 2$. Note that the function is undefined at 0, so the function is never equal to 2.



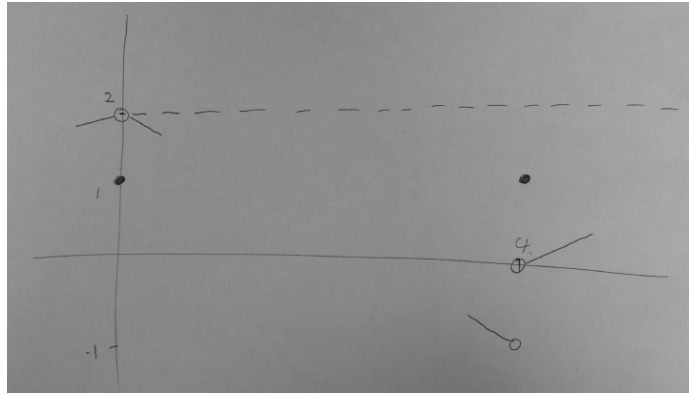
We can then add in the next condition $\lim_{t \rightarrow 4^-} g(t) = -1$



And repeat for the right sided limit:



Finally, we can add in the condition at $g(0) = g(4)$. Note that we have no constraints on them at the moment, so let's just define it to be at 1. Although any value other than 2 will work.



Combining all this information, we can guess that the graph looks like a straight line between $x = 0$ and $x = 4$ linking $(0, 2)$ and $(4, -1)$. This means line will have slope $m = \frac{-1-2}{4-0} = -\frac{3}{4}$ and y -intercept 2. Hence, we get the equation of the line segment to be $y = -\frac{3}{4}t + 2$. So our function will look like:

$$g(t) = \left\{ -\frac{3}{4}t + 2 \quad t \in (0, 4) \right.$$

To the left of 0, we must also have a function that is approaching 2 but never equal to 2. Any function of this form will work, but let's minimise our work by continuing the existing function to the left of $x = 0$ but not at $x = 0$ itself.

$$g(t) = \left\{ -\frac{3}{4}t + 2 \quad t \in (-\infty, 0) \cup (0, 4) \right.$$

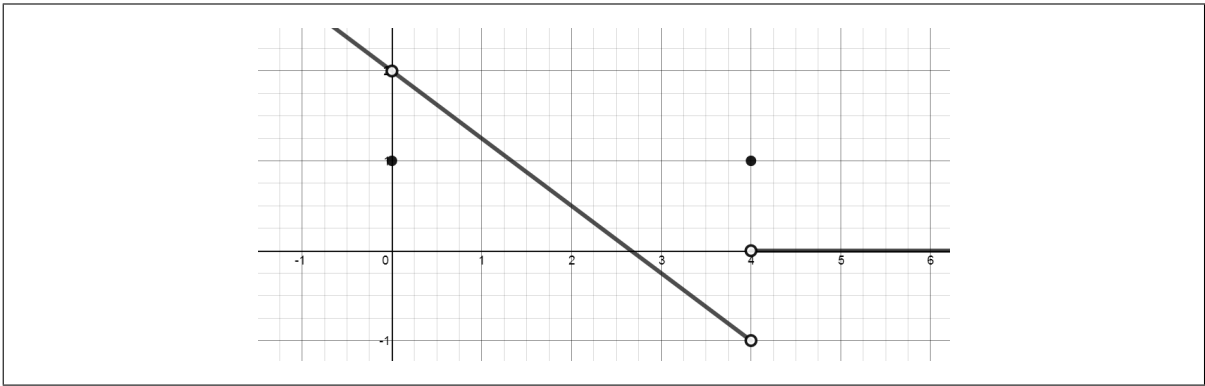
Now we have to consider what happens to the left of $x = 4$, the right sided limit is 0, so we need a function that gets close to 0 as x gets close to 4. The easiest such function is just $y = 0$. Again, any function with this property will work, provided that the function does not have 2 as an output anywhere to the right of 4. So we get:

$$g(t) = \begin{cases} -\frac{3}{4}t + 2 & t \in (-\infty, 0) \cup (0, 4) \\ 0 & t \in (4, \infty) \end{cases}$$

So now this takes care of every case except for $x = 0, 4$. So let's just put those in manually. I've made them equal to 1, but any point will work as long as it's not 2. So our final function will look like:

$$g(t) = \begin{cases} -\frac{3}{4}t + 2 & t \in (-\infty, 0) \cup (0, 4) \\ 0 & t \in (4, \infty) \\ 1 & t = 0, 4 \end{cases}$$

If we plot it using *Desmos* (an online calculator), we get something that looks like:



3. Construct a function $f(x)$ such that it satisfies $4 \leq f(x) \leq 8$ for all values of x , along with the condition that:

$$\lim_{x \rightarrow 0^-} f(x) = \text{Undefined}$$

$$\lim_{x \rightarrow 0^+} f(x) = \text{Undefined}$$

A full solution will require justification as to why the two one-sided limits do not exist for your function $f(x)$.

Solution: This is actually quite an interesting question (in my opinion). If we modify what was covered in the workshop, this question can be quite simple. Consider the function

$$f(x) = \begin{cases} 5 & x \in \mathbb{Q} \\ 7 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This basically means f will spit out 5 if the input is a rational number and it will spit out 7 if the input is an irrational number (ie: not a rational number). Using this function, let's look at the limit as $x \rightarrow 0^+$. If we start with $a = 4$, then we have $f(a) = 5$ cause 4 is rational. Then we consider a number closer to 0 than 4 but irrational (say $a = \pi$). Then we get $f(a) = 7$. Then we can try $a = \frac{4}{10}$, which is closer to 0 than π and rational. Hence it will output 5. Then we can try $a = \frac{\pi}{10}$ which is even closer to 0, and then try $a = \frac{4}{100}$ etc. Repeating this process, we can get an infinite sequence of numbers that keeps getting

close to 0 but with their output alternating between 5 and 7. For example:

$$\begin{array}{ll}
 a = 4 & f(4) = 5 \\
 a = \pi & f(\pi) = 7 \\
 a = \frac{4}{10} & f\left(\frac{4}{10}\right) = 5 \\
 a = \frac{\pi}{10} & f\left(\frac{\pi}{10}\right) = 7 \\
 a = \frac{4}{100} & f\left(\frac{4}{100}\right) = 5 \\
 a = \frac{\pi}{100} & f\left(\frac{\pi}{100}\right) = 7 \\
 a = \frac{4}{1000} & f\left(\frac{4}{1000}\right) = 5 \\
 a = \frac{\pi}{1000} & f\left(\frac{\pi}{1000}\right) = 7 \\
 a = \frac{4}{10000} & f\left(\frac{4}{10000}\right) = 5 \\
 a = \frac{\pi}{10000} & f\left(\frac{\pi}{10000}\right) = 7
 \end{array}$$

This means that we can keep picking numbers that get closer and closer to 0, but the output alternates between two numbers. So that means as we approach 0 from the positive side, we won't get to a unique number. Hence, the limit as $x \rightarrow 0^+$ cannot exist. The exact same argument can be made for $x \rightarrow 0^-$ using $a = -4$ and $a = -\pi$ and their fractions.

Alternate argument: We can argue that the function $f(x)$ we have picked has no one sided limit as follows: Take the numbers $a = 4, \frac{4}{10}, \frac{4}{100}, \frac{4}{1000}, \frac{4}{10000}, \dots$. We have sequence of numbers that get closer and closer to 0 all with output 5, which means our guess for the limit as $x \rightarrow 0^+$ would be 5. However, if we take an alternative sequence $a = \pi, \frac{\pi}{10}, \frac{\pi}{100}, \frac{\pi}{1000}, \frac{\pi}{10000}, \dots$, we also get a sequence that gets closer and close to 0 with outputs of 7. So our guess for the limit as $x \rightarrow 0^+$ would be 7. That means, we have two possible candidates for the limits depending on the numbers we take. This means we don't have a unique candidate and hence the one sided limit cannot exist (it doesn't go to a single number).

Note: The key thing here is that for a limit to exist, *ANY* sequence of numbers we take as we get close to 0 must produce outputs that get close to the same number. If the number that the output gets close to depends on our choice of x 's, then the limit does not exist.