1. The height of a ball at time t is $x(t) = t(4-t)$.

a) Use the definition of the derivative as a limit and interpretation of the velocity as the derivative of position with respect to time to find the velocity $v(t)$ of the ball at time t.

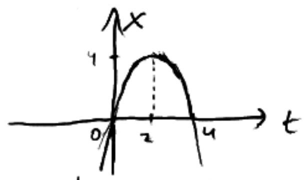
$$v(t) = x'(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h\to 0} \frac{(t+h)[4-(t+h)] - t(4-t)}{h} =$$

$$= \lim_{h\to 0} \frac{(t+h)(4-t-h) - t(4-t)}{h} = \lim_{h\to 0} \frac{(\cancel{4t} - \cancel{t^2} - ht + 4h - ht - h^2) - (\cancel{4t} - \cancel{t^2})}{h}$$

$$= \lim_{h\to 0} \frac{4h - 2ht - h^2}{h} = \lim_{h\to 0} (4 - 2t - h) = 4 - 2t$$

b) At what moment of time is the velocity equal to zero? At what moment of time is the height of the ball the largest? Is there any relation between these moments in time? If so, explain what it is and why it makes sense intuitively for such a relation to exist.

- $v(t) = 0$ when $4 - 2t = 0$, i.e. $t = 2$
- $x(t)$ is the largest when $t(4-t)$ is the largest. The graph of $x(t) = t(4-t) = 4t - t^2$ is a downward facing parabola:



The heighest point on this graph corresponds to $t = 2$.

- For $t < 2$ the velocity of the ball is positive, i.e. it moves up.
After $t = 2$ (i.e. for $t > 2$), it is negative, i.e. the ball moves down.
So the moment when $v(t) = 0$ corresponds to the time when the ball is at the heighest point.

2. We define the **dderivative** (that's right: a double d!) of the function f at point x to be the limit

$$f^{\nabla}(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

a) Let $f(x) = 2x + 3$. Find $f^{\nabla}(x)$.

$$f^{\nabla}(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h\to 0} \frac{[2(x+h)+3] - 2(2x+3) + [2(x-h)+3]}{h^2}$$

$$= \lim_{h\to 0} \frac{0}{h^2} = 0$$

b) Let $f(x) = ax + b$ for some unspecified numbers a, b. What is the dderivative of f?

$$f^{\nabla}(x) = \lim_{h\to 0} \frac{[a(x+h)+b] - 2(ax+b) + [a(x-h)+b]}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

c) Let $f(x) = x^2$. Find $f^{\nabla}(x)$.

$$f^{\nabla}(x) = \lim_{h\to 0} \frac{(x+h)^2 - 2x^2 + (x-h)^2}{h^2} = \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - 2x^2 + (x^2 - 2xh + h^2)}{h^2} =$$

$$= \lim_{h\to 0} \frac{2h^2}{h^2} = \lim_{h\to 0} 2 = 2$$

d) Find the dderivative of the function $f(x) = x^3$ at points $x = 1$ and $x = -1$.

$$f^\nabla(1) = \lim_{h\to 0} \frac{f(1+h) - 2f(1) + f(1-h)}{h^2} = \lim_{h\to 0} \frac{(1+h)^3 - 2\cdot 1^3 + (1-h)^3}{h^2} =$$

$$= \lim_{h\to 0} \frac{1+3h+3h^2+h^3 - 2 + 1 - 3h + 3h^2 - h^3}{h^2} = \lim_{h\to 0} \frac{6h^2}{h^2} = \lim_{h\to 0} 6 = 6$$

$$f^\nabla(-1) = \lim_{h\to 0} \frac{(-1+h)^3 - 2\cdot(-1)^3 + (-1-h)^3}{h^2} = \lim_{h\to 0} \frac{-1+3h-3h^2+h^3+2-1-3h-3h^2-h^3}{h^2} =$$

$$= \lim_{h\to 0} \frac{-6h^2}{h^2} = \lim_{h\to 0}(-6) = -6$$

(more generally $f^\nabla(x) = \lim_{h\to 0} \frac{(x+h)^3 - 2x^3 + (x-h)^3}{h^2} = \lim_{h\to 0} \frac{6xh^2}{h^2} = 6x$)

e) The dderivative of a function f at point x being **positive** means that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} > 0.$$

This means that for really small h the value of $\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ is positive, which in turn means that

$$\frac{f(x+h) + f(x-h)}{2} > f(x)$$

(you are encouraged to do the algebra on a separate sheet of paper to verify that this is indeed correct). Similarly $f^\nabla(x) < 0$ means that for really small h, the opposite inequality

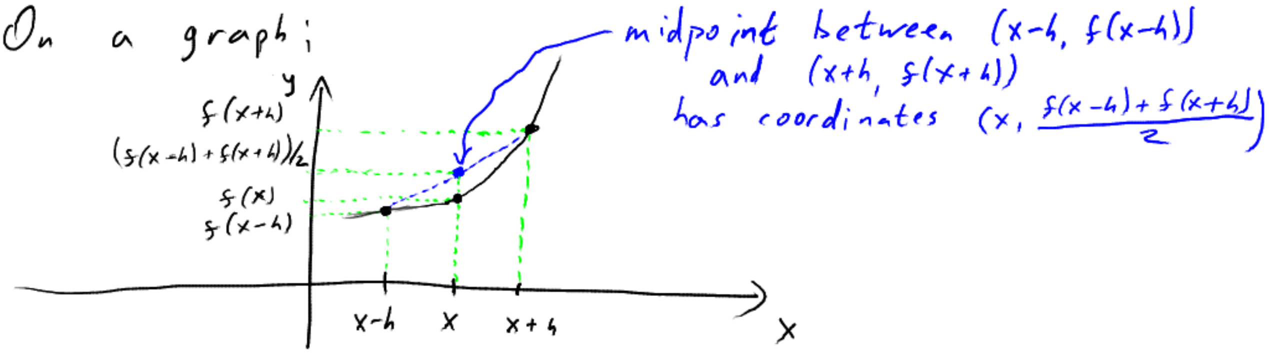
$$\frac{f(x+h) + f(x-h)}{2} < f(x)$$

holds.

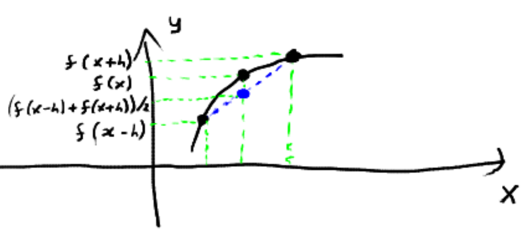
Think about what these inequalities mean on a graph of the function f. Based on this interpretation of the sign of the dderivative at point x, sketch three graphs: one graph of a function which is increasing and has positive dderivative at any point, one graph of a function that is decreasing and has positive dderivative at any point, and one graph of a function that is increasing and has a negative dderivative at any point. Label your graphs with labels "$f\nearrow, f^\nabla > 0$", "$f\searrow, f^\nabla > 0$" and "$f\nearrow, f^\nabla < 0$" respectively.

$\frac{f(x+h) + f(x-h)}{2} > f(x)$ means that $f(x)$ is smaller than the average of the values $f(x+h)$ and $f(x-h)$.

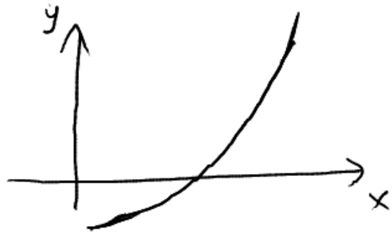
On a graph:



The opposite inequality $f(x) > \frac{f(x-h) + f(x+h)}{2}$ means that $f(x)$ is larger than the average of $f(x-h)$ and $f(x+h)$:



Requested graphs:



$f\nearrow$, $f^{\nabla} > 0$

$f\searrow$, $f^{\nabla} > 0$

$f\nearrow$, $f^{\nabla} < 0$

3. Let

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots \\ 0 & \text{otherwise} \end{cases}$$

a) Is f continuous at $x = 0$? Justify your answer.

f is continuous: $f(0)=0$ is defined

$\lim_{x\to 0} f(x) = 0$, because regardless how x approaches 0, the value of f is either equal to x (which gets close to zero as $x \to 0$) or to 0 (which is equal to zero).

So $f(0) = \lim_{x\to 0} f(x)$, so f is continuous at $x=0$.

(remark: but f is not continuous at $x = \frac{1}{10}$, $x = \frac{1}{100}$ and so on)

b) Does $f'(0)$ exist? Justify your answer. (Hint: use the definition of the derivative using limits).

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

$$\frac{f(h)}{h} = \begin{cases} 1, & \text{if } h = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots \\ \text{undefined} & \text{for } h = 0 \\ 0 & \text{otherwise} \end{cases}$$

As $h \to 0$, $\frac{f(h)}{h}$ can approach either 1 (if h approaches 0 along values $h = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ and so on)

or 0 (if h approaches 0 along values $h = \frac{1}{20}, \frac{1}{200}, \frac{1}{2000}$ and so on)

Hence $\lim_{h\to 0} \frac{f(h)}{h}$ does not exist, so

$f'(0)$ does not exist.

(Remark: for a similar looking function

$$g(x) = \begin{cases} x^2, & \text{if } x = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$g'(0)$ does exist and $g'(0) = 0$.

You are encouraged to think why this is true).