# ASSIGNMENT 1·9 (Section 002) Due: Friday, November 1

---

*There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions (in full sentences) with complete arguments and justifications in a linear, coherent manner. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. Your work must be your own and must be self-contained. Assignments must be stapled, with your name and student number at the top of each page. The assignment is due at the beginning of class on the due date.*

1. In class we proved the power rule for positive, integer exponents,

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{N} = \{1, 2, 3 \ldots\}.$$

Using the above and the quotient rule prove the power rule for negative, integer exponents. That is, show,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}, \quad n \in \mathbb{N}.$$

**Solution** We start with the left hand side and show it is equivalent to the right hand side. Note that $x^{-n} = \frac{1}{x^n}$ so we apply quotient rule.

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{0(x^n) - nx^{n-1}}{(x^n)^2},$$

where in the last step we have applied the power rule for positive exponents. Using exponent laws to clean our expression up yields,

$$\frac{d}{dx}(x^{-n}) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1-2n} = -nx^{-n-1},$$

the desired result. We have just shown that the power rule is valid for negative exponents.

2. In class we proved the product rule for differentiable functions $f$ and $g$,

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

(a) Using the product rule show that,

$$\frac{d}{dx}(fgh) = \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}.$$

(b) Using part (a) as well as the product rule show,

$$\frac{d}{dx}(f_1f_2f_3f_4) = \frac{df_1}{dx}f_2f_3f_4 + f_1\frac{df_2}{dx}f_3f_4 + f_1f_2\frac{df_3}{dx}f_4 + f_1f_2f_3\frac{df_4}{dx}.$$

**Solution** (a) We prove the above product rule for three functions by applying the traditional product rule for two functions. First we recognize $fgh$ as the product of two functions, namely $fg$ and $h$. In light of this we see,

$$\frac{d}{dx}\big(fgh\big) = \frac{d}{dx}\big((fg) \cdot h\big) = \frac{d}{dx}(fg) \cdot h + (fg) \cdot \frac{dh}{dx},$$

using the product rule. We again apply the product rule to expand the derivative of $(fg)$,

$$\frac{d}{dx}(fgh) = \left(\frac{df}{dx}g + f\frac{dg}{dx}\right)h + fg\frac{dh}{dx} = \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}.$$

We have arrived at the required result; a convenient formula for the derivative of a product of three functions.

(b) We consider the product of four functions. Observe that $f_1f_2f_3f_4 = (f_1f_2f_3)f_4$ a product of two functions. We take the derivative of a product of two functions by means of the product rule,

$$\frac{d}{dx}(f_1f_2f_3f_4) = \frac{d}{dx}\big((f_1f_2f_3)\cdot f_4\big) = \frac{d}{dx}(f_1f_2f_3)\cdot f_4 + (f_1f_2f_3)\frac{df_4}{dx}.$$

To expand the derivative of $f_1f_2f_3$ we appeal to the result from part (a). Upon substitution we see,

$$\begin{aligned}\frac{d}{dx}(f_1f_2f_3f_4) &= \left(\frac{df_1}{dx}f_2f_3 + f_1\frac{df_2}{dx}f_3 + f_1f_2\frac{df_3}{dx}\right)\cdot f_4 + f_1f_2f_3\frac{df_4}{dx} \\ &= \frac{df_1}{dx}f_2f_3f_4 + f_1\frac{df_2}{dx}f_3f_4 + f_1f_2\frac{df_3}{dx}f_4 + f_1f_2f_3\frac{df_4}{dx}.\end{aligned}$$

So, we have established the desired equality.
Note: We can prove a similar formula for the product of five functions by using product rule and appealing the the result from part (b). Moreover using the result for the product of five functions we can prove a result for the product of 6 functions. In fact, we can prove

$$\frac{d}{dx}(f_1\cdot f_2\cdot\ldots\cdot f_n) = \frac{df_1}{dx}f_2f_3\ldots f_n + f_1\frac{df_2}{dx}f_3f_4\ldots f_n + f_1f_2\frac{df_3}{dx}f_4\ldots f_n + \ldots + f_1f_2f_3\ldots f_{n-1}\frac{df_n}{dx}$$

for any integer $n$ (no matter how large) using a technique known as Mathematical Induction.

3. (a) Consider a differential function $g(x)$. Define,

$$f(x) = xg(x).$$

Compute $f'(0)$ in terms of function $g$.

(b) Let

$$f(x) = x\frac{(x+1)}{2}\frac{(x+2)}{3}\frac{(x+3)}{4}\cdots\frac{(x+99)}{100}.$$

Compute $f'(0)$ for this function $f$.

**Solution** (a) We apply product rule to find the derivative of $f$. Observe, $f'(x) = g(x) + xg'(x)$. We can substitute zero to obtain: $f'(0) = g(0) + 0\cdot g'(0)$ or rather $f'(0) = g(0)$. Thus we have computed $f'(0)$ in terms of function $g$.
(b) Let

$$g(x) = \frac{(x+1)}{2}\frac{(x+2)}{3}\frac{(x+3)}{4}\cdots\frac{(x+99)}{100}.$$

In this way, $f(x) = xg(x)$. Using the result from part (a) we know that $f'(0) = g(0)$. It is left to compute this quantity. Notice in

$$g(0) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{99}{100},$$

that the 2's will cancel, the 3's will cancel, the 4's will cancel and so on until the 99's cancel. In light of this we are left with $g(0) = 1/100$ and so $f'(0) = 1/100$.