There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.

1. *In this problem we will see that not all related rates problems involve time.* You own a factory that is making a length of wire out of two metals: copper and iron. You have access to mines containing both iron and copper. The amount A you are able to produce is determined by the equation

$$A = 10000\sqrt{xy}$$

where x is the number of tonnes of copper mined and y is the number of tonnes of iron mined. However, you only have a limited number of mining resources. The amount of copper and iron you can mine is determined by the following relationship

$$x^2 + 4xy + 15y^2 = 144.$$

If we mine more copper then we are forced to mine less iron. In this way we can think of y as a function of x. The quantity dA/dx represents the change in the amount of wire we can produce as we change the amount of copper.

(a) Find dA/dx as a function of x and y.

(b) Find the value of dA/dx when the amount of copper mined is equal to the amount of iron mined. Is it positive or negative? Should you mine more or less copper than this amount?

Solution (a) We start by differentiating $A = 10000x^{1/2}y^{1/2}$ with respect to x

$$\frac{dA}{dx} = 10000\left(\frac{1}{2}x^{-1/2}y^{1/2} + \frac{1}{2}x^{1/2}y^{-1/2}\frac{dy}{dx}\right)$$

$$\frac{dA}{dx} = 5000\left(\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}}\frac{dy}{dx}\right).$$

We must eliminate dy/dx from the equation. We can find this value by differentiating the second equation implicitly

$$2x + 4y + 4x\frac{dy}{dx} + 30y\frac{dy}{dx} = 0$$

$$x + 2y + 2x\frac{dy}{dx} + 15y\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 15y\frac{dy}{dx} = -x - 2y$$

$$(2x + 15y)\frac{dy}{dx} = -x - 2y$$

$$\frac{dy}{dx} = -\frac{x + 2y}{2x + 15y}.$$

Substituting to our equation for dA/dx gives the desired quantity as a function of only x and y

$$\frac{dA}{dx} = 5000\left(\sqrt{\frac{y}{x}} - \sqrt{\frac{x}{y}}\frac{x + 2y}{2x + 15y}\right).$$

(b) We can find when the values of copper and iron are equal. Set $x = y$ and substitute to our constraint equation

$$x^2 + 4x^2 + 15x^2 = 144.$$

So we can now solve for x

$$20x^2 = 144$$
$$x^2 = \frac{144}{20}$$
$$x = \sqrt{\frac{144}{20}}$$
$$x = \frac{12}{\sqrt{20}}$$
$$x = \frac{12}{2\sqrt{2}}$$
$$x = \frac{6}{\sqrt{2}}.$$

We have selected only the positive root since we mine only positive amounts of copper. With this value in hand we can substitute $x = y = 6/\sqrt{2}$ to our equation for dA/dx

$$\frac{dA}{dx} = 5000\left(\sqrt{\frac{y}{x}} - \sqrt{\frac{x}{y}}\,\frac{x + 2y}{2x + 15y}\right)$$
$$= 5000\left(\sqrt{\frac{x}{x}} - \sqrt{\frac{x}{x}}\,\frac{x + 2x}{2x + 15x}\right)$$
$$= 5000\left(\sqrt{\frac{x}{x}} - \sqrt{\frac{x}{x}}\,\frac{3x}{17x}\right)$$
$$= 5000\left(1 - 1 \cdot \frac{3}{17}\right)$$
$$= 5000\left(\frac{14}{17}\right) \approx 4117.647 > 0.$$

It turns out we can solve for this value of dA/dx without finding the x where $x = y$. The derivative dA/dx is positive at this point. This means the the amount function $A(x)$ is increasing. So increasing x will increase the total amount of wire that is produced. Thus we should increase the amount of copper mined.

2. Out in the wild west a man has been injured in a duel. He steadies himself against a light post that is three times as tall as he is. He summons his strength and begins to walk in a straight line away from the light post. He accelerates initially but soon slows. After making it 6 meters he stops to catch his breath. His speed v (measured in meters per second) throughout his journey is described by

$$v = -\frac{s^2}{4} + \frac{3s}{2}$$

where s is his distance from the light post in meters.

(a) Find the rate of change of the length of his shadow when he has traveled 4 meters from the light post.

(b) Show that his shadow was changing the fastest when he reached his top speed.

Solution

(a) We start by drawing a picture. Let h be the height of the man, let s be his distance from the light post and let x be the length of his shadow. See figure. We have an expression for $ds/dt = v(s)$ and desire an expression for dx/dt.
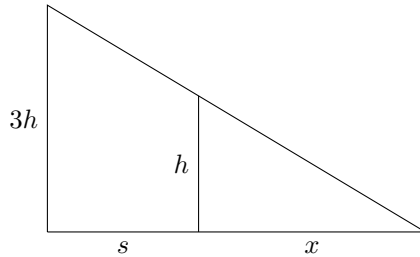
Using similar triangles we see

$$\frac{s+x}{3h} = \frac{x}{h}.$$

After cross multiplying the $h's$ cancel and we see

$$s + x = 3x$$
$$s = 2x$$
$$x = \frac{s}{2}.$$

With our equation relating s and x we can now differentiate with respect to time

$$\frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt}.$$

We have an equation for $v(s) = ds/dt$ and can substitute it to find an expression for dx/dt as a function of s his distance from the post

$$\frac{dx}{dt} = \frac{1}{2}\left(-\frac{s^2}{4} + \frac{3s}{2}\right).$$

We seek dx/dt when $s = 4$ and can see this after substitution

$$\frac{dx}{dt} = \frac{1}{2}\left(-\frac{4^2}{4} + \frac{3 \cdot 4}{2}\right)$$
$$= \frac{1}{2}\left(-\frac{4^2}{4} + \frac{3 \cdot 4}{2}\right)$$
$$= \frac{1}{2}\left(-4 + 3 \cdot 2\right)$$
$$= -2 + 3 = 1\text{m/s}$$

(b) If we consider the equation

$$\frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt}$$

obtained in part (a) we see that the max of his speed ds/dt and the max rate of change of the length of his shadow dx/dt must occur at the same time. The rate of change of the length of his shadow is equal to his speed divided by two. Dividing a function by two does not change where the global max occurs and so the two have the same max.

Alternatively we can consider the function $v(s) = -s^2/4 + 3s/2$ on the domain $s \in [0, 6]$ and find the max using the closed interval method. At the endpoints we have $v(0) = 0$ and $v(6) = 0$. The lone critical point is $s = 3$ and $v(3) = 2.25 > 0$. So the man reaches his max speed when his distance is $x = 3$. Now we can also consider the function

$$f(s) = \frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}\left(-\frac{s^2}{4} + \frac{3s}{2}\right)$$

and find its max on $[0, 6]$. Using the closed interval method yields a critical point $s = 3$ which is the global max. Hence the two maximums occur at the same point in time.

3. You stand 5 meters underneath a pulley. You hold in your hand a rope that loops up through the pulley and is connected to a heavy box of calculus exams which sits on the ground exactly 12 meters behind you. The rope is stretched tight at all times. You rush forward at 2 meters per second pulling the rope with you. How fast is the box moving after 2.5 seconds have elapsed? Include any assumptions you make to solve the problem.

   Solution Initially the system looks like this
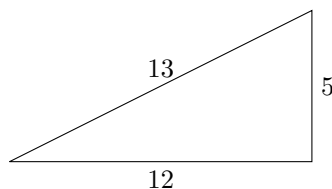
   where we have found the hypotenuse by means of the Pythagorean Theorem. This assumes that the heights of you and box are about the same. We could run the same argument giving you height, say h, and replace 5 in the above as $5 + h$. This problem will not be any harder. Note that from the above we recognize that the rope has fixed length $L = 13 + 5 = 18$. When we put the system into motion we see the following
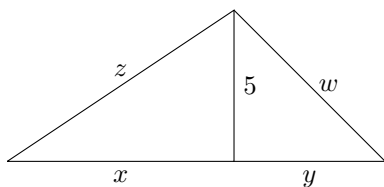
   where x is the distance from the box to directly beneath the pulley, y is the distance from you to directly beneath the pulley, z is the length of rope connecting the box to the pulley and w is the length of rope connecting you to the pulley. We desire some equations relating these quantities. The Pythagorean Theorem gives us

$$x^2 + 5^2 = z^2$$
$$y^2 + 5^2 = w^2$$

   and the fact that the rope has fixed total length $L = 18$ gives

$$z + w = 18.$$

   Let us solve for z and w in the first two equations and then substitute to the third equation

$$z = \sqrt{x^2 + 25}$$
$$w = \sqrt{y^2 + 25}$$

so we have

$$\sqrt{x^2 + 25} + \sqrt{y^2 + 25} = 18.$$

With an equation relating x and y in hand we are prepared to take the derivative with respect to time

$$0 = \frac{1}{2}\left(x^2 + 25\right)^{-1/2}\left(2x\frac{dx}{dt}\right) + \frac{1}{2}\left(y^2 + 25\right)^{-1/2}\left(2y\frac{dy}{dt}\right)$$

$$0 = \frac{x}{\sqrt{x^2 + 25}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 25}}\frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2 + 25}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 25}}\frac{dy}{dt}$$

So we can now solve for dx/dt

$$\frac{x}{\sqrt{x^2 + 25}}\frac{dx}{dt} = -\frac{y}{\sqrt{y^2 + 25}}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y\sqrt{x^2 + 25}}{x\sqrt{y^2 + 25}}\frac{dy}{dt}.$$

We must find x and y after 2.5 seconds. Note that your speed $dy/dt = 2$ and so your distance after 2.5 seconds is $y = 5$ using that distance is speed times time. Now to find x. After 2.5 seconds we have $w = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$. The fixed length of the rope gives $z = 18 - 5\sqrt{2}$. Another application of the Pythagorean Theorem gives

$$x = \sqrt{(18 - 5\sqrt{2})^2 - 5^2} = \ldots = \sqrt{349 - 180\sqrt{2}} \approx 9.7181$$
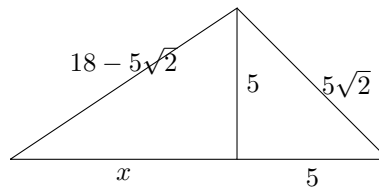
Putting everything together gives

$$\frac{dx}{dt} = -\frac{5\sqrt{349 - 180\sqrt{2} + 25}}{\sqrt{349 - 180\sqrt{2}}\sqrt{5^2 + 25}}(2) \approx -1.59\text{m/s}.$$

Notice however that this is the rate at which the box's distance from directly below the pulley is changing. This distance is shrinking and so we see a negative value. The forward speed of the box is given by the positive value 1.59 m/s.