

Questions:

1. The *yawn factor* is a measure of the ability for an instructor to put their students to sleep during a lecture. The relationship between the yawn factor y of an instructor and the number of students who attends lectures s is given by:

$$\frac{1920}{y \cdot s} = \sqrt{y + s}$$

- (a) Validate that a yawn factor of 4 will result in an attendance of 60 students.

Solution: Validating a solution is the easy part. We just check that the point $y = 4$ and $s = 60$ is on the given curve. Checking both side separately:

$$\begin{aligned} \text{LHS} &= \frac{1920}{y \cdot s} \\ &= \frac{1920}{4 \cdot 60} \\ &= 8 \\ \text{RHS} &= \sqrt{y + s} \\ &= \sqrt{60 + 4} \\ &= 8 \end{aligned}$$

So both sides are equal and the point is on the curve. We're good to go.

- (b) Use a linear approximation to estimate the attendance if the instructor works really hard to decrease their yawn factor to 3.

Solution: This is the main part of the question. Since the variables are symmetric, we can assume $y(s)$ or $s(y)$ and do either linear approximation with equal difficulty. But since we will be altering y later, it makes sense to choose $s(y)$. Recall our linear approximation is give by:

$$L(y) = s(a) + s'(a)(y - a)$$

where a is our nice point. Based on this setup, we will choose $a = 4$ since we know

when $y = 4$, $s(4) = 60$. Now we need $s'(4)$. Tackling this implicitly:

$$\begin{aligned}\frac{d}{dy} \left(\frac{1920}{y \cdot s} \right) &= \frac{d}{dy} (\sqrt{y+s}) \\ 1920 \frac{d}{dy} ((y \cdot s)^{-1}) &= \frac{1 + \frac{ds}{dy}}{2\sqrt{y+s}} \\ 1920 \left((-1)(y \frac{ds}{dy} + s)(y \cdot s)^{-2} \right) &= \frac{1 + \frac{ds}{dy}}{2\sqrt{y+s}} \\ \frac{-1920 \cdot y \frac{ds}{dy}}{(y \cdot s)^2} + \frac{-1920 \cdot s}{(y \cdot s)^2} &= \frac{\frac{ds}{dy}}{2\sqrt{y+s}} + \frac{1}{2\sqrt{y+s}} \\ \frac{-1920 \cdot y \frac{ds}{dy}}{(y \cdot s)^2} - \frac{\frac{ds}{dy}}{2\sqrt{y+s}} &= \frac{1}{2\sqrt{y+s}} + \frac{1920 \cdot s}{(y \cdot s)^2} \\ \frac{ds}{dy} \left(\frac{-1920 \cdot y}{(y \cdot s)^2} - \frac{1}{2\sqrt{y+s}} \right) &= \frac{1}{2\sqrt{y+s}} + \frac{1920 \cdot s}{(y \cdot s)^2} \\ \frac{ds}{dy} &= \frac{\frac{1}{2\sqrt{y+s}} + \frac{1920 \cdot s}{(y \cdot s)^2}}{\left(\frac{-1920 \cdot y}{(y \cdot s)^2} - \frac{1}{2\sqrt{y+s}} \right)}\end{aligned}$$

Now we substitute $y = 4$ and $s = 60$

$$\begin{aligned}\frac{ds}{dy} &= \frac{\frac{1}{2\sqrt{(4)+(60)}} + \frac{1920 \cdot (60)}{((4) \cdot (60))^2}}{\left(\frac{-1920 \cdot (4)}{((4) \cdot (60))^2} - \frac{1}{2\sqrt{(4)+(60)}} \right)} \\ &= \frac{\frac{1}{2\sqrt{64}} + \frac{1920}{(4) \cdot (60)} \cdot \frac{60}{(4) \cdot (60)}}{\left(\frac{1920}{(4) \cdot (60)} \cdot \frac{-4}{(4) \cdot (60)} - \frac{1}{2\sqrt{64}} \right)} \\ &= \frac{\frac{1}{16} + 8 \cdot \frac{1}{4}}{\left(8 \cdot \frac{-1}{60} - \frac{1}{16} \right)} \\ &= \frac{\frac{1}{16} + 2}{\left(\frac{-2}{15} - \frac{1}{16} \right)} \\ &= -\frac{\frac{33}{16}}{\frac{47}{240}} \\ \frac{ds}{dy} &= -\frac{495}{47}\end{aligned}$$

Now that we have the derivative at the point, we can apply our linear approximation.

$$\begin{aligned}L(y) &= s(a) + s'(a)(y - a) \\ L(y) &= 60 - \frac{495}{47}(y - 4)\end{aligned}$$

At $y = 3$, we get:

$$\begin{aligned}L(3) &= 60 - \frac{495}{47}(3 - 4) \\ &= 60 + \frac{495}{47} \\ &= \frac{3315}{47} \approx 70.53.\end{aligned}$$

Optimistically speaking, the instructor should expect about 71 students to attend if their yawn factor is decreased by one. *For completeness, if we get a computer to solve this, we get a true value of $s = 73.2787$. Not a horrible estimate.*

2. Consider the function:

$$g(x) = e^x \sin(x)$$

- (a) Find the linear approximation $L(x)$ around $a = 0$ and use it to approximate $g(1)$.

Solution: We use the formula to find the equation of the tangent line to $g(x)$ around $a = 0$

$$L(x) = g(0) + g'(0)(x - 0)$$

Note that $g(0) = e^0 \sin(0) = 0$. We compute the derivative

$$g'(x) = e^x \sin x + e^x \cos x$$

so $g'(0) = e^0 \sin(0) + e^0 \cos(0) = 1$. Our equation now reads

$$L(x) = x.$$

We now use this line to approximate the desired value: $g(1) \approx L(1) = 1$.

- (b) Find the second order approximation $T_2(x)$ around $a = 0$ and use it to approximate $g(1)$. That is let $T_2(x) = L(x) + bx^2$ and choose b so that $g''(0) = T_2''(0)$.

Solution: To improve our approximation we can use a quadratic function rather than a line. Let $T_2(x) = L(x) + bx^2$ so

$$T_2(x) = x + bx^2.$$

We will find the constant b so that $g''(0) = T_2''(0)$. We compute

$$\begin{aligned} g''(x) &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x \\ &= 2e^x \cos x. \end{aligned}$$

So $g''(0) = 2$. As well

$$T_2'(x) = 1 + 2bx$$

$$T_2''(x) = 2b$$

Solving $2 = 2b$ gives $b = 1$. Therefore

$$T_2(x) = 1 + x^2.$$

Using this function to approximate gives: $g(1) \approx T_2(1) = 2$.

- (c) Find the third order approximation $T_3(x)$ around $a = 0$ and use it to approximate $g(1)$. That is let $T_3(x) = T_2(x) + cx^3$ and choose c so that $g'''(0) = T_3'''(0)$.

Solution: Again we compute some derivatives

$$g'''(x) = 2e^x \cos x - 2e^x \sin x.$$

So $g'''(0) = 2$. As well

$$T_3(x) = x + x^2 + cx^3$$

$$T_3'(x) = 1 + 2x + 3cx^2$$

$$T_3''(x) = 2 + 6cx$$

$$T_3'''(x) = 6c.$$

Requiring $g'''(0) = T_3'''(0)$ gives $6c = 2$ or rather $c = 1/3$. Hence

$$T_3(x) = x + x^2 + \frac{1}{3}x^3.$$

Using this function gives the approximation: $g(1) \approx T_3(1) = 2 + \frac{1}{3} = 2.333$.

- (d) Compare these values to the calculator given value of $g(1)$.

Solution: The calculator gives a value of $g(1) = e^1 \sin(1) = 2.287$. As we can see each new approximation is performing better than the last. To achieve more than one decimal place we would take $T_4(x)$ or $T_5(x)$ or $T_{10}(x)$ or higher depending on how many decimal places we wanted.

3. For this question we are interested in the following function:

$$f(x) = \sin(x) - x^2$$

- (a) Show that the function has at least one critical point.

Solution: If we want to show that the function has at least one critical point, we want to see when the derivative is undefined or equal to zero. Looking at the derivative:

$$f'(x) = \cos(x) - 2x$$

Each term in the derivative is continuous and hence there are no discontinuities. Therefore, we want to show that the derivative has at least one root. This is screaming out IVT. So let's do it. We know the function is continuous, so now we just need a point above and below zero.

$$f'(0) = 1 > 0$$

$$f'\left(\frac{\pi}{2}\right) = -\pi < 0$$

So there must be a root in $f'(x)$ between $(0, \frac{\pi}{2})$. This means there is a critical point in $f(x)$ over the same interval.

- (b) Use Newton's Method to find a critical point (numerically) accurate to 6 digits and justify how you know your answer is accurate to 6 digits.

Solution: Now that we know there's a root somewhere, we can start hunting for it. For a function $h(x)$ and a initial guess x_0 , we have:

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$$

In this example, we are looking for roots in $f'(x)$, so we will use $h(x) = f'(x)$. This means we need $h'(x) = f''(x) = -\sin(x) - 2$. Since we know the root is close to 0, let's just start with $x_0 = 0$.

$$\begin{aligned} x_1 &= x_0 - \frac{\cos(x_0) - 2x_0}{-\sin(x_0) - 2} \\ &= 0 - \frac{\cos(0) - 2(0)}{-\sin(0) - 2} \\ &= 0.5 \\ x_2 &= x_1 - \frac{\cos(x_1) - 2x_1}{-\sin(x_1) - 2} \\ &= 0 - \frac{\cos(0.5) - 2(0.5)}{-\sin(0.5) - 2} \\ &= 0.45062669307724304657 \\ x_3 &= x_2 - \frac{\cos(x_2) - 2x_2}{-\sin(x_2) - 2} \\ &= 0.45018364757777474250 \end{aligned}$$

Summarizing all the numbers in a table, we get:

n	x_n
0	0
1	0.5
2	0.450626693077243
3	0.450183647577775
4	0.450183611294874
5	0.450183611294874

We know we can stop iterating once the next few digits stop changing. Note that this only took 5 iterations to get 14 digits of accuracy. Compare that to the bisection method we used in Assignment 1.6 where it took about 15 iterations to get 3 digits of accuracy. This function is actually pretty nicely behaved. Just because we can, suppose we take a terrible first guess of $x_0 = 10000$ which is very far from the root. This is what the iteration looks like:

n	x_n
0	10000
1	-1804.25047533596
2	1276.273715539467
3	334.0464410033218
4	100.7228888065848
5	9.21799844502284
6	0.414382592767333
7	0.4504264606881685
8	0.4501836221959263
9	0.4501836112948736
10	0.4501836112948736

Just a final note: This is why (existence) theorems like IVT and MVT are useful. Once we have established that what we are interested in actually exists, we can use numerical methods such as Newton's Method to go hunting for them.
