Instructions:

- There are two parts to this assignment. The first part is on WeBWorK. Submit answers online.
- The second parts consists of questions on this page. The solutions must be typed.
- A physical copy of this assignment is due at the start of class (8am) on the date specified above.
- You are expected to provide full, justified solutions for each question.
- The solutions will be graded on the correctness, clarity and elegance.
- Please pay close attention to the **assignment policy**.

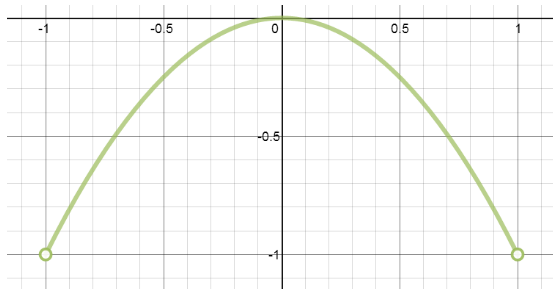
Questions:

1. For each of the following, construct a function that satisfies the given condition. That is, sketch a graph and provide the corresponding algebraic expression.

   (a) A continuous function on an open interval with a global max and no global min.

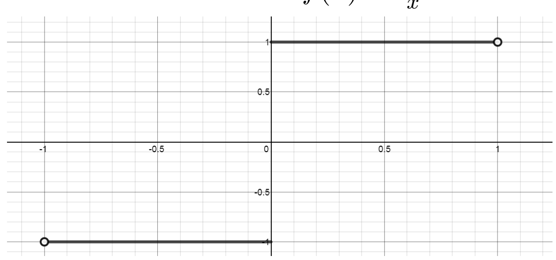
Solution: Note: With many parts of this question, there are literally infintely many solution. All we have to do is find one that works. I will state the function and provide the graph. But realistically, it was easier to come up with the graph first and then come by with the algebraic function.

Simplest solution I can think of: $f(x) = -x^2$ on the interval $(-1, 1)$.



   (b) A discontinuous function on an open interval with a global max and global min.

Solution: Simplest solution I can think of: $f(x) = \frac{|x|}{x}$ on the interval $(-1, 1)$.



Note: Technically, every point in that function is a global max or a global min. But if you feel somewhat cheated by that, I offer two more solutions:

$$f(x) = \begin{cases} \sin(x) & x \in (-\pi, 0) \cup (0, \pi) \\ 1 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in (-2, -1) \\ x & x \in [-1, 1] \\ 0 & x \in (1, 2) \end{cases}$$

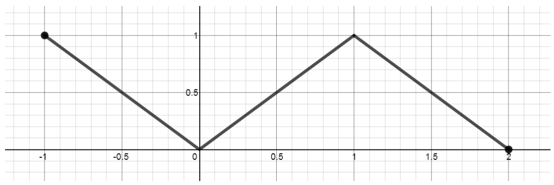
Since this is an open interval, the global max and min cannot occur at the endpoints. So that means we have to include the critical points (and hence the extrema) inside the interval.

(c) A function on a close interval where the global max and global min occurs at points where the derivative is not defined.

Solution: Simplest solution I can think of:

$$f(x) = \begin{cases} |x| & x \in [-1, 1] \\ 2 - x & x \in (1, 2] \end{cases}$$

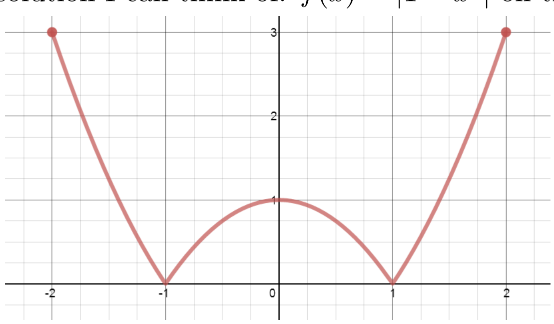
on the interval $[-1, 2]$.



This one is a little more difficult. In order for the derivative to not be defined, we need the function with a break in the derivative function. The absolute value function will satisfy one of the extrema. We then just conjure the other one to make it work. Technically, we can use something like $f(x) = 0$ on the interval $[-1, 1]$ and say that the max and min happens at the endpoints where the derivatives are not defined, but I feel that's bending the rules a little too much.

(d) A continuous function with both types of critical points (zero derivative and a derivative that does not exist).

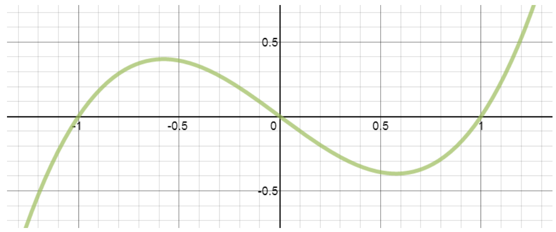
Solution: Simplest solution I can think of: $f(x) = |1 - x^2|$ on the interval $[-2, 2]$.



To get this, we need a function where the derivative does not exist. The absolute value function is the simplest such function we have. Then we just have to use a function with a zero derivative somewhere off the x axis.

(e) A function with exactly two critical points, but no global max and no global min.

Solution: Simplest solution I can think of: $f(x) = x(x-1)(x+1)$ on the real numbers.



We want a function where the derivative has two points of zero derivative. The cubic function is the easiest such function because it's derivative is a quadratic and hence has at most two solutions. To ensure the lack of global max and min, we need an open interval where the tails are higher and lower than the critical points.

2. Let $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

   (a) Find all the global maxima of the function $f(x)$ on the interval $\left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$. Justify your answer.

Solution: To find the global maxima of the function, we need to look at the critical points of the function. So that means we have to find the points where the derivative is either 0 or undefined. Let's start by looking at the first part of the piecewise function and finding the derivative of that bit:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \sin\left(\frac{1}{x}\right) &= \cos\left(\frac{1}{x}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) \\ &= \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{aligned}$$

Since this is a fraction, we can check when the derivative fails to exist by seeing when the denominator is zero, We obtain that $x = 0$ as the only solution, but this is not in the domain of the first piece, so we can ignore it (for now). Next, we want to find when the derivative is zero by setting the numerator to zero. This reduces the problem to finding solving

$$\cos\left(\frac{1}{x}\right) = 0.$$

We can solve this by first looking at the values of d such that $\cos(d) = 0$, which we know how tackle. We get that

$$d = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Since we have $d = \frac{1}{x}$, we know that $\cos\left(\frac{1}{x}\right) = 0$ whenever

$$x = \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \pm\frac{2}{5\pi}, \pm\frac{2}{7\pi} \ldots.$$

Next, we should determine which of the critical points are within the interval we're interested in. Since the fractions have the same numerator and increasing denominators, we can conclude the that fractions are indeed decreasing. Combined with the fact that first point is in the interval, we get that all the critical points lists must be in the interval. Yes, we have a function on a closed interval with infinitely many critical points. Yikes.

Now that we have the critical points, let's see which ones are maxima and which ones are not. Lets consider the positive x values first:

$$f\left(\frac{2}{\pi}\right) = 1$$
$$f\left(\frac{2}{3\pi}\right) = -1$$
$$f\left(\frac{2}{5\pi}\right) = 1$$
$$f\left(\frac{2}{7\pi}\right) = -1$$

Looking at this, we expect the pattern to repeat (since it's a sine function) and so we get that global maximum to be 1 when $x > 0$. Similarly, with the negative x values:

$$f\left(-\frac{2}{\pi}\right) = -1$$
$$f\left(-\frac{2}{3\pi}\right) = 1$$
$$f\left(-\frac{2}{5\pi}\right) = -1$$
$$f\left(-\frac{2}{7\pi}\right) = 1$$

Comparing the postive and negative x values, we can conclude that the global maximum is 1 which happens at:

$$x = \frac{2}{\pi}, -\frac{2}{3\pi}, \frac{2}{5\pi}, -\frac{2}{7\pi}, \ldots$$

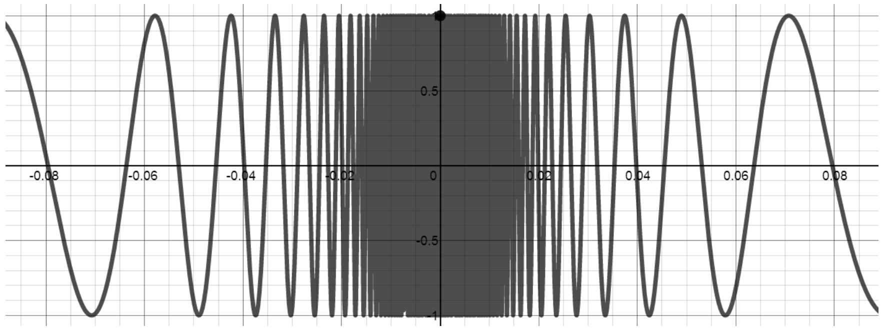
with the next one occurring when we increment the denominator by 2π and flipping the sign. Finally, since the global maximum is 1, we have to include $x = 0$ since $f(0) = 1$. So the list of x values where the maximum occurs is:

$$x = 0, \frac{2}{\pi}, -\frac{2}{3\pi}, \frac{2}{5\pi}, -\frac{2}{7\pi}, \ldots$$

(b) [Bonus] Is $f(x)$ continuous? Why or why not?

Solution: This function $f(x)$ is not a function we've seen previously, but we can check its continuity by apply the definition. When $x \neq 0$, we are in the first case and since $\frac{1}{x}$ and $\sin(x)$ are both continuous functions when $x \neq 0$, their composition will also be a continuous function. So that only point we have to deal with is $x = 0$. First, we see that $f(0) = 1$ and hence $f(x)$ is defined at $x = 0$. Next, we want to find the limit as x

goes to 0. This limit exists if the two one sided limits exists and are equal. Next, let's take a look at a plot of the function to get a better idea.



Sure, it looks like a mess. So our intuition might tell us that the limit of something like that will not exist. Let's see if we can prove it. From our points in the previous part, we can choose something like:

$$x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi}, \frac{2}{17\pi}, \ldots$$

and evaluating $f(x)$ at those points will all give 1. So that would suggest

$$\lim_{x \to 0^+} f(x) = 1$$

. However, if we try another sequence of x values, such as:

$$x = \frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi}, \frac{2}{15\pi}, \frac{2}{19\pi}, \ldots$$

and evaluating $f(x)$ at those points, we will get -1. So that would suggest

$$\lim_{x \to 0^+} f(x) = -1$$

. But comparing the two sequences, we see that we have two candidates for the left hand limit. Recall from last term that when we have more than one candidate for a limit, that means the one sided limit cannot exist. So that means

$$\lim_{x \to 0^+} f(x) \text{ DNE.}$$

Since the one sided limits do not exist, the overall limit cannot exist and therefore the function cannot be continuous at $x = 0$. Thus, the function is not continuous at all real numbers.

3. In the 1998 paper "Comparing Gompertz and Richards Functions to Estimate Freezing Injury in *Rhododendron* Using Electrolyte Leakage", published in the *Journal of the American Society for Horticultural Science*, the authors propose a function which models the effect of temperature on the "injury level" of rhododendron leaves. The function is

$$f(T) = 100e^{-be^{kT}},$$

where b and k are positive constants and T is temperature (in degrees Celcius).

(a) The function $f'(T)$ has exactly one critical point. Find that point.

Solution: Since we're looking for critical points, we will have to see where the derivative is either zero or does not exist. Either way, we need to find the derivative (either through chain rule or log-differentiation). Let:

$$\begin{aligned} A(T) &= kT & A'(T) &= k \\ B(A) &= -be^A & B'(A) &= -be^A \\ f(B) &= 100e^B & f'(B) &= 100e^B \end{aligned}$$

So using the chain rule, we get:

$$\begin{aligned} f'(T) &= f'(B) \cdot B'(A) \cdot A'(T) \\ &= 100e^B \cdot \left(-be^A\right) \cdot (K) \\ &= 100e^{-be^{kT}} \cdot \left(-be^{kT}\right) \cdot (K) \\ &= -100bke^{-be^{kT}+kT} \end{aligned}$$

We are looking at critical points, we want to find the points where the derivative is either zero or undefined. Since, the function composes of embedded exponential functions and everything is defined. So we only have to find the values of T for which the derivative is zero. Looking at the function, we see that we have a bunch of positive constant multiplied by the exponential function. That means the derivative $f'(T)$ will never be equal to zero on the domain.

Umm... So we look at the question again and we see that we are actually looking for the critical points of $f'(T)$ and not the critical points of $F(T)$, which is what we've found. To find the critical points of the derivative, we need the second derivative. So let us at it:

$$\begin{aligned} f'(T) &= -100bke^{-be^{kT}+kT} \\ f''(T) &= \frac{\mathrm{d}}{\mathrm{d}T} - 100bke^{-be^{kT}+kT} \\ &= -100bk\frac{\mathrm{d}}{\mathrm{d}T}e^{-be^{kT}+kT} && \text{Const. Multiple} \\ &= -100bk \cdot \left(e^{-be^{kT}+kT}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}T}\left(-be^{kT}+kT\right) && \text{Chain} \\ &= -100bk \cdot \left(e^{-be^{kT}+kT}\right) \cdot \left(-bke^{kT}+k\right) \\ &= \left(-100bk^2\right) \cdot \left(e^{-be^{kT}+kT}\right) \cdot \left(-be^{kT}+1\right) \end{aligned}$$

The second derivative looks like a mess but thankfully, we are looking for the critical points, so we are looking for $F''(T) = 0$. Luckily, the function is factored nicely and we know that if several functions multiply to zero, then at least one of them has to be zero.

- $-100bk^2 \neq 0$ since it is a bunch of non-zero constants.
- $e^{-be^{kT}+kT} \neq 0$ since the output of an exponential function is never zero.

- $1 - be^{kT} = 0$ could happen, so let's see if we can find it.

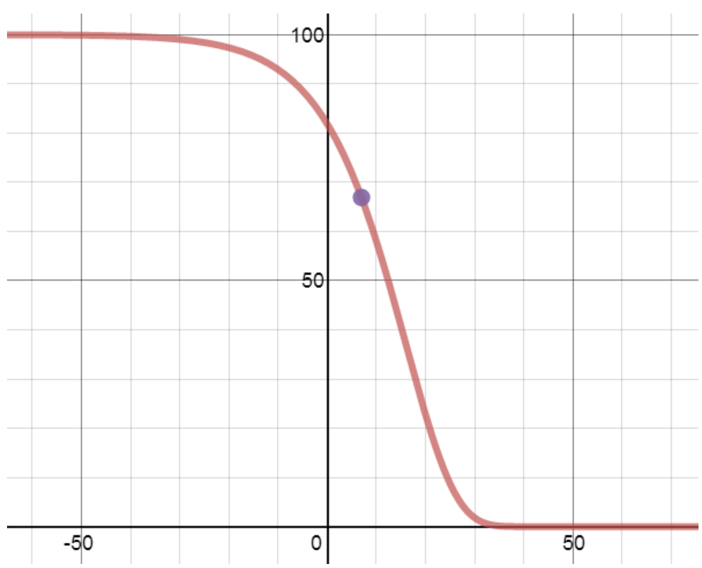
We want to solve for T in the last case:

$$\begin{aligned} 1 - be^{kT} &= 0 \\ 1 &= be^{kT} \\ \log(1) &= \log(be^{kT}) \\ 0 &= \log(b) + \log(e^{kT}) \\ 0 &= \log(b) + kT \\ -\log(b) &= kT \\ T &= \frac{-\log(b)}{k} \end{aligned}$$

Since the parameters b, k are non-zero, we get a value for T. So that means the single critical point of $f'(T)$ happens at $T = \frac{-\log(b)}{k}$.

(b) What might that critical point represent? Propose an explanation in a few sentences. Your answer should refer both to the function $f'(T)$ and the physical phenomenon being described. You may use a picture if you wish.

Solution: Since this is a critical point on the derivative, we might expect that this temperature to be a maximum or a minimum *on the derivative function*. This really means that the slope of the original function has a maximum or a minimum slope at that particular temperature. We can validate that by plotting an example of the function (For me, I used $b = 0.2$, $k = 0.1$).



Looking at that graph, it seems to agree with our intuition that the purple dot appears to have the steepest slope. That is, the graph has most negative slope at the critical point we found in the previous part. Translating this into the context of the question: The temperature we found in the previous part corresponds to the temperature at which the "injury levels" of rhododendron leaves the quickest.